39. Suppose that $\lim_{h\to 0} g(h) = L$.

(a) Explain why $\lim_{h\to 0} g(ah) = L$ for any constant $a \neq 0$.

(b) If we assume instead that $\lim_{h\to 1} g(h) = L$, is it still necessarily true that $\lim_{h\to 1} g(ah) = L$?

(c) Illustrate (a) and (b) with the function $f(x) = x^2$.

SOLUTION

(a) As $h \to 0$, $ah \to 0$ as well; hence, if we make the change of variable $w = ah$, then

$$\lim_{h\to 0} g(ah) = \lim_{w\to 0} g(w) = L.$$

(b) No. As $h \to 1$, $ah \to a$, so we should not expect $\lim_{h\to 1} g(ah) = \lim_{h\to 1} g(h)$.

(c) Let $g(x) = x^2$. Then

$$\lim_{h\to 0} g(h) = 0 \quad \text{and} \quad \lim_{h\to 0} g(ah) = \lim_{h\to 0} (ah)^2 = 0.$$

On the other hand,

$$\lim_{h\to 1} g(h) = 1 \quad \text{while} \quad \lim_{h\to 1} g(ah) = \lim_{h\to 1} (ah)^2 = a^2,$$

which is equal to the previous limit if and only if $a = \pm 1$.

40. Assume that $L(a) = \lim_{x\to 0} \dfrac{a^x - 1}{x}$ exists for all $a > 0$. Assume also that $\lim_{x\to 0} a^x = 1$.

(a) Prove that $L(ab) = L(a) + L(b)$ for $a, b > 0$. *Hint:* $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$. This shows that $L(a)$ "behaves" like a logarithm. We will see that $L(a) = \ln a$ in Section 3.10.

(b) Verify numerically that $L(12) = L(3) + L(4)$.

SOLUTION

(a) Let $a, b > 0$. Then

$$\begin{aligned} L(ab) = \lim_{x\to 0} \frac{(ab)^x - 1}{x} &= \lim_{x\to 0} \frac{a^x(b^x - 1) + (a^x - 1)}{x} \\ &= \lim_{x\to 0} a^x \cdot \lim_{x\to 0} \frac{b^x - 1}{x} + \lim_{x\to 0} \frac{a^x - 1}{x} \\ &= 1 \cdot L(b) + L(a) = L(a) + L(b). \end{aligned}$$

(b) From the table below, we estimate that, to three decimal places, $L(3) = 1.099$, $L(4) = 1.386$ and $L(12) = 2.485$. Thus,

$$L(12) = 2.485 = 1.099 + 1.386 = L(3) + L(4).$$

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$(3^x - 1)/x$	1.092600	1.098009	1.098552	1.098673	1.099216	1.104669
$(4^x - 1)/x$	1.376730	1.385334	1.386198	1.386390	1.387256	1.395948
$(12^x - 1)/x$	2.454287	2.481822	2.484600	2.485215	2.488000	2.516038

2.4 Limits and Continuity

Preliminary Questions

1. Which property of $f(x) = x^3$ allows us to conclude that $\lim_{x\to 2} x^3 = 8$?

SOLUTION We can conclude that $\lim_{x\to 2} x^3 = 8$ because the function x^3 is continuous at $x = 2$.

2. What can be said about $f(3)$ if f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$?

SOLUTION If f is continuous and $\lim_{x\to 3} f(x) = \frac{1}{2}$, then $f(3) = \frac{1}{2}$.

3. Suppose that $f(x) < 0$ if x is positive and $f(x) > 1$ if x is negative. Can f be continuous at $x = 0$?

SOLUTION Since $f(x) < 0$ when x is positive and $f(x) > 1$ when x is negative, it follows that

$$\lim_{x\to 0+} f(x) \le 0 \quad \text{and} \quad \lim_{x\to 0-} f(x) \ge 1.$$

Thus, $\lim_{x\to 0} f(x)$ does not exist, so f cannot be continuous at $x = 0$.

4. Is it possible to determine $f(7)$ if $f(x) = 3$ for all $x < 7$ and f is right-continuous at $x = 7$? What if f is left-continuous?

SOLUTION No. To determine $f(7)$, we need to combine either knowledge of the values of $f(x)$ for $x < 7$ with *left*-continuity or knowledge of the values of $f(x)$ for $x > 7$ with right-continuity.

5. Are the following true or false? If false, state a correct version.

(a) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and are equal.

(b) $f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$.

(c) If the left- and right-hand limits of $f(x)$ as $x \to a$ exist, then f has a removable discontinuity at $x = a$.

(d) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

(e) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)/g(x)$ is continuous at $x = a$.

SOLUTION

(a) False. The correct statement is "$f(x)$ is continuous at $x = a$ if the left- and right-hand limits of $f(x)$ as $x \to a$ exist and equal $f(a)$."

(b) True.

(c) False. The correct statement is "If the left- and right-hand limits of $f(x)$ as $x \to a$ are equal but not equal to $f(a)$, then f has a removable discontinuity at $x = a$."

(d) True.

(e) False. The correct statement is "If $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$."

Exercises

1. Referring to Figure 1, state whether $f(x)$ is left- or right-continuous (or neither) at each point of discontinuity. Does $f(x)$ have any removable discontinuities?

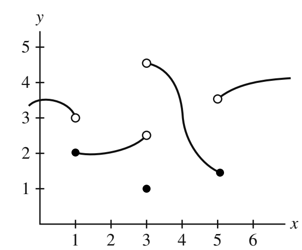


FIGURE 1 Graph of $y = f(x)$

SOLUTION

- The function f is discontinuous at $x = 1$; it is right-continuous there.
- The function f is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function f is discontinuous at $x = 5$; it is left-continuous there.

However, these discontinuities are not removable.

Exercises 2–4 refer to the function $g(x)$ in Figure 2.

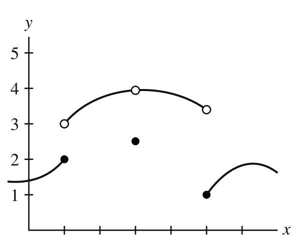


FIGURE 2 Graph of $y = g(x)$

2. State whether $g(x)$ is left- or right-continuous (or neither) at each of its points of discontinuity.

SOLUTION

- The function g is discontinuous at $x = 1$; it is left-continuous there.
- The function g is discontinuous at $x = 3$; it is neither left-continuous nor right-continuous there.
- The function g is discontinuous at $x = 5$; it is right-continuous there.

3. At which point c does $g(x)$ have a removable discontinuity? How should $g(c)$ be redefined to make g continuous at $x = c$?

SOLUTION Because $\lim_{x\to 3} g(x)$ exists, the function g has a removable discontinuity at $x = 3$. Assigning $g(3) = 4$ makes g continuous at $x = 3$.

4. Find the point c_1 at which $g(x)$ has a jump discontinuity but is left-continuous. How should $g(c_1)$ be redefined to make g right-continuous at $x = c_1$?

SOLUTION The function g has a jump discontinuity at $x = 1$, but is left-continuous there. Assigning $g(1) = 3$ makes g right-continuous at $x = 1$ (but no longer left-continuous).

5. In Figure 3, determine the one-sided limits at the points of discontinuity. Which discontinuity is removable and how should f be redefined to make it continuous at this point?

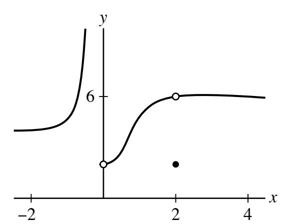


FIGURE 3

SOLUTION The function f is discontinuous at $x = 0$, at which $\lim_{x\to 0-} f(x) = \infty$ and $\lim_{x\to 0+} f(x) = 2$. The function f is also discontinuous at $x = 2$, at which $\lim_{x\to 2-} f(x) = 6$ and $\lim_{x\to 2+} f(x) = 6$. Because the two one-sided limits exist and are equal at $x = 2$, the discontinuity at $x = 2$ is removable. Assigning $f(2) = 6$ makes f continuous at $x = 2$.

6. Suppose that $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

(a) What is $f(3)$ if f is left-continuous at $x = 3$?

(b) What is $f(3)$ if f is right-continuous at $x = 3$?

SOLUTION $f(x) = 2$ for $x < 3$ and $f(x) = -4$ for $x > 3$.

- If f is left-continuous at $x = 3$, then $f(3) = \lim_{x\to 3-} f(x) = 2$.
- If f is right-continuous at $x = 3$, then $f(3) = \lim_{x\to 0+} f(x) = -4$.

In Exercises 7–16, use the Laws of Continuity and Theorems 2 and 3 to show that the function is continuous.

7. $f(x) = x + \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x + \sin x$ by Continuity Law (i).

8. $f(x) = x \sin x$

SOLUTION Since x and $\sin x$ are continuous, so is $x \sin x$ by Continuity Law (iii).

9. $f(x) = 3x + 4\sin x$

SOLUTION Since x and $\sin x$ are continuous, so are $3x$ and $4\sin x$ by Continuity Law (ii). Thus $3x + 4\sin x$ is continuous by Continuity Law (i).

10. $f(x) = 3x^3 + 8x^2 - 20x$

SOLUTION

- Since x is continuous, so are x^3 and x^2 by repeated applications of Continuity Law (iii).
- Hence $3x^3$, $8x^2$, and $-20x$ are continuous by Continuity Law (ii).
- Finally, $3x^3 + 8x^2 - 20x$ is continuous by Continuity Law (i).

11. $f(x) = \dfrac{1}{x^2 + 1}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Recall that constant functions, such as 1, are continuous. Thus $x^2 + 1$ is continuous.
- Finally, $\dfrac{1}{x^2+1}$ is continuous by Continuity Law (iv) because $x^2 + 1$ is never 0.

12. $f(x) = \dfrac{x^2 - \cos x}{3 + \cos x}$

SOLUTION

- Since x is continuous, so is x^2 by Continuity Law (iii).
- Since $\cos x$ is continuous, so is $-\cos x$ by Continuity Law (ii).
- Accordingly, $x^2 - \cos x$ is continuous by Continuity Law (i).
- Since 3 (a constant function) and $\cos x$ are continuous, so is $3 + \cos x$ by Continuity Law (i).
- Finally, $\dfrac{x^2 - \cos x}{3 + \cos x}$ is continuous by Continuity Law (iv) because $3 + \cos x$ is never 0.

13. $f(x) = \cos(x^2)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\cos x$ and x^2, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

14. $f(x) = \tan^{-1}(4^x)$

SOLUTION The function $f(x)$ is a composite of two continuous functions: $\tan^{-1} x$ and 4^x, so $f(x)$ is continuous by Theorem 5, which states that a composite of continuous functions is continuous.

15. $f(x) = e^x \cos 3x$

SOLUTION e^x and $\cos 3x$ are continuous, so $e^x \cos 3x$ is continuous by Continuity Law (iii).

16. $f(x) = \ln(x^4 + 1)$

SOLUTION

- Since x is continuous, so is x^4 by repeated application of Continuity Law (iii).
- Since 1 (a constant function) and x^4 are continuous, so is $x^4 + 1$ by Continuity Law (i).
- Finally, because $x^4 + 1 > 0$ for all x and $\ln x$ is continuous for $x > 0$, the composite function $\ln(x^4 + 1)$ is continuous.

In Exercises 17–34, determine the points of discontinuity. State the type of discontinuity (removable, jump, infinite, or none of these) and whether the function is left- or right-continuous.

17. $f(x) = \dfrac{1}{x}$

SOLUTION The function $1/x$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

18. $f(x) = |x|$

SOLUTION The function $f(x) = |x|$ is continuous everywhere.

19. $f(x) = \dfrac{x-2}{|x-1|}$

SOLUTION The function $\dfrac{x-2}{|x-1|}$ is discontinuous at $x = 1$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 1$.

20. $f(x) = [x]$

SOLUTION This function has a jump discontinuity at $x = n$ for every integer n. It is continuous at all other values of x. For every integer n,

$$\lim_{x \to n+} [x] = n$$

since $[x] = n$ for all x between n and $n + 1$. This shows that $[x]$ is *right-continuous* at $x = n$. On the other hand,

$$\lim_{x \to n-} [x] = n - 1$$

since $[x] = n - 1$ for all x between $n - 1$ and n. Thus $[x]$ is not left-continuous.

21. $f(x) = \left[\frac{1}{2}x\right]$

SOLUTION The function $\left[\frac{1}{2}x\right]$ is discontinuous at even integers, at which there are jump discontinuities. Because

$$\lim_{x \to 2n+} \left[\frac{1}{2}x\right] = n$$

but

$$\lim_{x \to 2n-} \left[\frac{1}{2}x\right] = n - 1,$$

it follows that this function is right-continuous at the even integers but not left-continuous.

22. $g(t) = \dfrac{1}{t^2 - 1}$

SOLUTION The function $f(t) = \dfrac{1}{t^2 - 1} = \dfrac{1}{(t-1)(t+1)}$ is discontinuous at $t = -1$ and $t = 1$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

23. $f(x) = \dfrac{x+1}{4x-2}$

SOLUTION The function $f(x) = \dfrac{x+1}{4x-2}$ is discontinuous at $x = \frac{1}{2}$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = \frac{1}{2}$.

24. $h(z) = \dfrac{1 - 2z}{z^2 - z - 6}$

SOLUTION The function $f(z) = \dfrac{1-2z}{z^2 - z - 6} = \dfrac{1-2z}{(z+2)(z-3)}$ is discontinuous at $z = -2$ and $z = 3$, at which there are infinite discontinuities. The function is neither left- nor right- continuous at either point of discontinuity.

25. $f(x) = 3x^{2/3} - 9x^3$

SOLUTION The function $f(x) = 3x^{2/3} - 9x^3$ is defined and continuous for all x.

26. $g(t) = 3t^{-2/3} - 9t^3$

SOLUTION The function $g(t) = 3t^{-2/3} - 9t^3$ is discontinuous at $t = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $t = 0$.

27. $f(x) = \begin{cases} \dfrac{x-2}{|x-2|} & x \neq 2 \\ -1 & x = 2 \end{cases}$

SOLUTION For $x > 2$, $f(x) = \dfrac{x-2}{(x-2)} = 1$. For $x < 2$, $f(x) = \dfrac{(x-2)}{(2-x)} = -1$. The function has a jump discontinuity at $x = 2$. Because

$$\lim_{x \to 2-} f(x) = -1 = f(2)$$

but

$$\lim_{x \to 2+} f(x) = 1 \neq f(2),$$

it follows that this function is left-continuous at $x = 2$ but not right-continuous.

28. $f(x) = \begin{cases} \cos \dfrac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

SOLUTION The function $\cos\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$, at which there is an oscillatory discontinuity. Because neither

$$\lim_{x \to 0-} f(x) \quad \text{nor} \quad \lim_{x \to 0+} f(x)$$

exist, the function is neither left- nor right-continuous at $x = 0$.

29. $g(t) = \tan 2t$

SOLUTION The function $g(t) = \tan 2t = \dfrac{\sin 2t}{\cos 2t}$ is discontinuous whenever $\cos 2t = 0$; i.e., whenever

$$2t = \frac{(2n+1)\pi}{2} \qquad \text{or} \qquad t = \frac{(2n+1)\pi}{4},$$

where n is an integer. At every such value of t there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

30. $f(x) = \csc(x^2)$

SOLUTION The function $f(x) = \csc(x^2) = \dfrac{1}{\sin(x^2)}$ is discontinuous whenever $\sin(x^2) = 0$; i.e., whenever $x^2 = n\pi$ or $x = \pm\sqrt{n\pi}$, where n is a positive integer. At every such value of x there is an infinite discontinuity. The function is neither left- nor right-continuous at any of these points of discontinuity.

31. $f(x) = \tan(\sin x)$

SOLUTION The function $f(x) = \tan(\sin x)$ is continuous everywhere. Reason: $\sin x$ is continuous everywhere and $\tan u$ is continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$—and in particular on $-1 \le u = \sin x \le 1$. Continuity of $\tan(\sin x)$ follows by the continuity of composite functions.

32. $f(x) = \cos(\pi[x])$

SOLUTION The function $f(x) = \cos(\pi[x])$ has a jump discontinuity at $x = n$ for every integer n. The function is right-continuous but not left-continuous at each of these points of discontinuity.

33. $f(x) = \dfrac{1}{e^x - e^{-x}}$

SOLUTION The function $f(x) = \dfrac{1}{e^x - e^{-x}}$ is discontinuous at $x = 0$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 0$.

34. $f(x) = \ln|x - 4|$

SOLUTION The function $f(x) = \ln|x - 4|$ is discontinuous at $x = 4$, at which there is an infinite discontinuity. The function is neither left- nor right-continuous at $x = 4$.

In Exercises 35–48, determine the domain of the function and prove that it is continuous on its domain using the Laws of Continuity and the facts quoted in this section.

35. $f(x) = 2\sin x + 3\cos x$

SOLUTION The domain of $2\sin x + 3\cos x$ is all real numbers. Both $\sin x$ and $\cos x$ are continuous on this domain, so $2\sin x + 3\cos x$ is continuous by Continuity Laws (i) and (ii).

36. $f(x) = \sqrt{x^2 + 9}$

SOLUTION The domain of $\sqrt{x^2 + 9}$ is all real numbers, as $x^2 + 9 > 0$ for all x. Since $\sqrt{x}$ and the polynomial $x^2 + 9$ are both continuous, so is the composite function $\sqrt{x^2 + 9}$.

37. $f(x) = \sqrt{x}\sin x$

SOLUTION This function is defined as long as $x \ge 0$. Since $\sqrt{x}$ and $\sin x$ are continuous, so is $\sqrt{x}\sin x$ by Continuity Law (iii).

38. $f(x) = \dfrac{x^2}{x + x^{1/4}}$

SOLUTION This function is defined as long as $x \ge 0$ and $x + x^{1/4} \ne 0$, and so the domain is all $x > 0$. Since x is continuous, so are x^2 and $x + x^{1/4}$ by Continuity Laws (iii) and (i); hence, by Continuity Law (iv), so is $\dfrac{x^2}{x + x^{1/4}}$.

39. $f(x) = x^{2/3}2^x$

SOLUTION The domain of $x^{2/3}2^x$ is all real numbers as the denominator of the rational exponent is odd. Both $x^{2/3}$ and 2^x are continuous on this domain, so $x^{2/3}2^x$ is continuous by Continuity Law (iii).

40. $f(x) = x^{1/3} + x^{3/4}$

SOLUTION The domain of $x^{1/3} + x^{3/4}$ is $x \ge 0$. On this domain, both $x^{1/3}$ and $x^{3/4}$ are continuous, so $x^{1/3} + x^{3/4}$ is continuous by Continuity Law (i).

41. $f(x) = x^{-4/3}$

SOLUTION This function is defined for all $x \ne 0$. Because the function $x^{4/3}$ is continuous and not equal to zero for $x \ne 0$, it follows that

$$x^{-4/3} = \frac{1}{x^{4/3}}$$

is continuous for $x \ne 0$ by Continuity Law (iv).

42. $f(x) = \ln(9 - x^2)$

SOLUTION The domain of $\ln(9 - x^2)$ is all x such that $9 - x^2 > 0$, or $|x| < 3$. The polynomial $9 - x^2$ is continuous for all real numbers and $\ln x$ is continuous for $x > 0$; therefore, the composite function $\ln(9 - x^2)$ is continuous for $|x| < 3$.

43. $f(x) = \tan^2 x$

SOLUTION The domain of $\tan^2 x$ is all $x \neq \pm(2n-1)\pi/2$ where n is a positive integer. Because $\tan x$ is continuous on this domain, it follows from Continuity Law (iii) that $\tan^2 x$ is also continuous on this domain.

44. $f(x) = \cos(2^x)$

SOLUTION The domain of $\cos(2^x)$ is all real numbers. Because the functions $\cos x$ and 2^x are continuous on this domain, so is the composite function $\cos(2^x)$.

45. $f(x) = (x^4+1)^{3/2}$

SOLUTION The domain of $(x^4+1)^{3/2}$ is all real numbers as $x^4+1>0$ for all x. Because $x^{3/2}$ and the polynomial x^4+1 are both continuous, so is the composite function $(x^4+1)^{3/2}$.

46. $f(x) = e^{-x^2}$

SOLUTION The domain of e^{-x^2} is all real numbers. Because e^x and the polynomial $-x^2$ are both continuous for all real numbers, so is the composite function e^{-x^2}.

47. $f(x) = \dfrac{\cos(x^2)}{x^2-1}$

SOLUTION The domain for this function is all $x \neq \pm 1$. Because the functions $\cos x$ and x^2 are continuous on this domain, so is the composite function $\cos(x^2)$. Finally, because the polynomial x^2-1 is continuous and not equal to zero for $x \neq \pm 1$, the function $\dfrac{\cos(x^2)}{x^2-1}$ is continuous by Continuity Law (iv).

48. $f(x) = 9^{\tan x}$

SOLUTION The domain of $9^{\tan x}$ is all $x \neq \pm(2n-1)\pi/2$ where n is a positive integer. Because $\tan x$ and 9^x are continuous on this domain, it follows that the composite function $9^{\tan x}$ is also continuous on this domain.

49. Show that the function

$$f(x) = \begin{cases} x^2+3 & \text{for } x<1 \\ 10-x & \text{for } 1 \le x \le 2 \\ 6x-x^2 & \text{for } x>2 \end{cases}$$

is continuous for $x \neq 1, 2$. Then compute the right- and left-hand limits at $x = 1, 2$, and determine whether $f(x)$ is left-continuous, right-continuous, or continuous at these points (Figure 4).

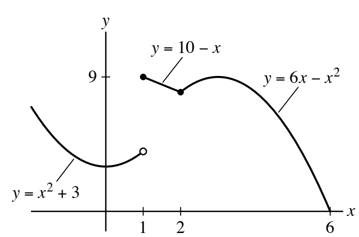


FIGURE 4

SOLUTION Let's start with $x \neq 1, 2$.

- Because x is continuous, so is x^2 by Continuity Law (iii). The constant function 3 is also continuous, so x^2+3 is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $x<1$.
- Because x and the constant function 10 are continuous, the function $10-x$ is continuous by Continuity Law (i). Therefore, $f(x)$ is continuous for $1<x<2$.
- Because x is continuous, x^2 is continuous by Continuity Law (iii) and $6x$ is continuous by Continuity Law (ii). Therefore, $6x-x^2$ is continuous by Continuity Law (i), so $f(x)$ is continuous for $x>2$.

At $x=1$, $f(x)$ has a jump discontinuity because the one-sided limits exist but are not equal:

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (x^2+3) = 4, \qquad \lim_{x\to 1+} f(x) = \lim_{x\to 1+} (10-x) = 9.$$

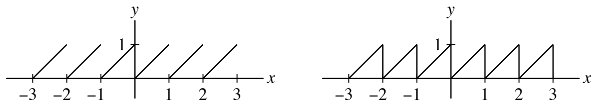
Furthermore, the right-hand limit equals the function value $f(1)=9$, so $f(x)$ is right-continuous at $x=1$. At $x=2$,

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} (10-x) = 8, \qquad \lim_{x\to 2+} f(x) = \lim_{x\to 2+} (6x-x^2) = 8.$$

The left- and right-hand limits exist and are equal to $f(2)$, so $f(x)$ is continuous at $x=2$.

50. Sawtooth Function Draw the graph of $f(x) = x - [x]$. At which points is f discontinuous? Is it left- or right-continuous at those points?

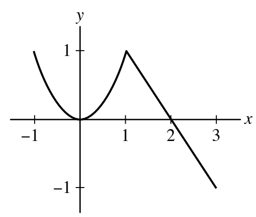
SOLUTION Two views of the sawtooth function $f(x) = x - [x]$ appear below. The first is the actual graph. In the second, the jumps are "connected" so as to better illustrate its "sawtooth" nature. The function is right-continuous at integer values of x.



In Exercises 51–54, sketch the graph of $f(x)$. At each point of discontinuity, state whether f is left- or right-continuous.

51. $f(x) = \begin{cases} x^2 & \text{for } x \le 1 \\ 2 - x & \text{for } x > 1 \end{cases}$

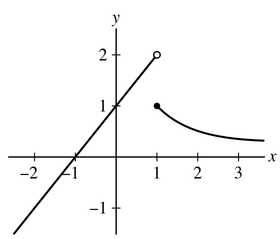
SOLUTION



The function f is continuous everywhere.

52. $f(x) = \begin{cases} x + 1 & \text{for } x < 1 \\ \dfrac{1}{x} & \text{for } x \ge 1 \end{cases}$

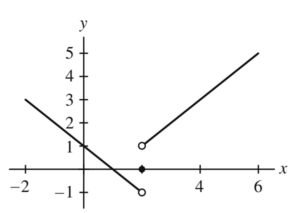
SOLUTION



The function f is right-continuous at $x = 1$.

53. $f(x) = \begin{cases} \dfrac{x^2 - 3x + 2}{|x - 2|} & x \ne 2 \\ 0 & x = 2 \end{cases}$

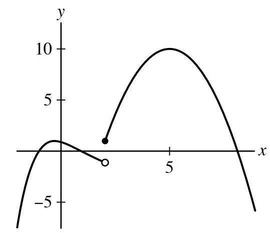
SOLUTION



The function f is neither left- nor right-continuous at $x = 2$.

54. $f(x) = \begin{cases} x^3 + 1 & \text{for } -\infty < x \le 0 \\ -x + 1 & \text{for } 0 < x < 2 \\ -x^2 + 10x - 15 & \text{for } x \ge 2 \end{cases}$

SOLUTION



The function f is right-continuous at $x = 2$.

55. Show that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 16}{x - 4} & x \neq 4 \\ 10 & x = 4 \end{cases}$$

has a removable discontinuity at $x = 4$.

SOLUTION To show that $f(x)$ has a removable discontinuity at $x = 4$, we must establish that

$$\lim_{x \to 4} f(x)$$

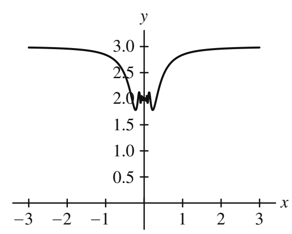
exists but does not equal $f(4)$. Now,

$$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} (x + 4) = 8 \neq 10 = f(4);$$

thus, $f(x)$ has a removable discontinuity at $x = 4$. To remove the discontinuity, we must redefine $f(4) = 8$.

56. GU Define $f(x) = x \sin \frac{1}{x} + 2$ for $x \neq 0$. Plot $f(x)$. How should $f(0)$ be defined so that f is continuous at $x = 0$?

SOLUTION



From the graph, it appears that $f(0)$ should be defined equal to 2 to make f continuous at $x = 0$.

In Exercises 57–59, find the value of the constant (a, b, or c) that makes the function continuous.

57. $f(x) = \begin{cases} x^2 - c & \text{for } x < 5 \\ 4x + 2c & \text{for } x \geq 5 \end{cases}$

SOLUTION As $x \to 5-$, we have $x^2 - c \to 25 - c = L$. As $x \to 5+$, we have $4x + 2c \to 20 + 2c = R$. Match the limits: $L = R$ or $25 - c = 20 + 2c$ implies $c = \frac{5}{3}$.

58. $f(x) = \begin{cases} 2x + 9x^{-1} & \text{for } x \leq 3 \\ -4x + c & \text{for } x > 3 \end{cases}$

SOLUTION As $x \to 3-$, we have $2x + 9x^{-1} \to 9 = L$. As $x \to 3+$, we have $-4x + c \to c - 12 = R$. Match the limits: $L = R$ or $9 = c - 12$ implies $c = 21$.

59. $f(x) = \begin{cases} x^{-1} & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x \leq \frac{1}{2} \\ x^{-1} & \text{for } x > \frac{1}{2} \end{cases}$

SOLUTION As $x \to -1-$, $x^{-1} \to -1$ while as $x \to -1+$, $ax + b \to b - a$. For f to be continuous at $x = -1$, we must therefore have $b - a = -1$. Now, as $x \to \frac{1}{2}-$, $ax + b \to \frac{1}{2}a + b$ while as $x \to \frac{1}{2}+$, $x^{-1} \to 2$. For f to be continuous at $x = \frac{1}{2}$, we must therefore have $\frac{1}{2}a + b = 2$. Solving these two equations for a and b yields $a = 2$ and $b = 1$.

60. Define

$$g(x) = \begin{cases} x+3 & \text{for } x < -1 \\ cx & \text{for } -1 \le x \le 2 \\ x+2 & \text{for } x > 2 \end{cases}$$

Find a value of c such that $g(x)$ is

(a) left-continuous

(b) right-continuous

In each case, sketch the graph of $g(x)$.

SOLUTION

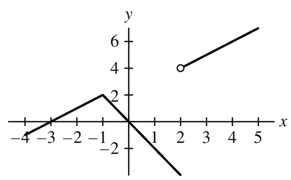
(a) In order for $g(x)$ to be left-continuous, we need

$$\lim_{x\to -1-} g(x) = \lim_{x\to -1-} (x+3) = 2$$

to be equal to

$$\lim_{x\to -1+} g(x) = \lim_{x\to -1+} cx = -c.$$

Therefore, we must have $c = -2$. The graph of $g(x)$ with $c = -2$ is shown below.



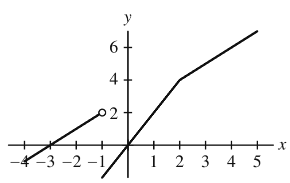
(b) In order for $g(x)$ to be right-continuous, we need

$$\lim_{x\to 2-} g(x) = \lim_{x\to 2-} cx = 2c$$

to be equal to

$$\lim_{x\to 2+} g(x) = \lim_{x\to 2+} (x+2) = 4.$$

Therefore, we must have $c = 2$. The graph of $g(x)$ with $c = 2$ is shown below.



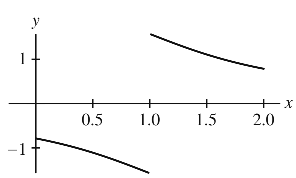
61. Define $g(t) = \tan^{-1}\left(\dfrac{1}{t-1}\right)$ for $t \neq 1$. Answer the following questions, using a plot if necessary.

(a) Can $g(1)$ be defined so that $g(t)$ is continuous at $t = 1$?

(b) How should $g(1)$ be defined so that $g(t)$ is left-continuous at $t = 1$?

SOLUTION

(a) From the graph of $g(t)$ shown below, we see that g has a jump discontinuity at $t = 1$; therefore, $g(a)$ cannot be defined so that g is continuous at $t = 1$.



(b) To make g left-continuous at $t = 1$, we should define

$$g(1) = \lim_{t\to 1-} \tan^{-1}\left(\tfrac{1}{t-1}\right) = -\frac{\pi}{2}.$$

62. Each of the following statements is *false*. For each statement, sketch the graph of a function that provides a counterexample.

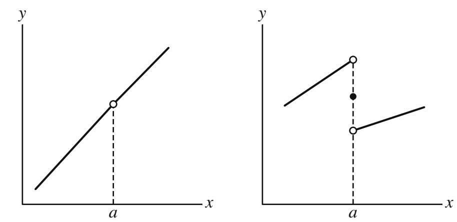
(a) If $\lim_{x\to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

(b) If $f(x)$ has a jump discontinuity at $x = a$, then $f(a)$ is equal to either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a+} f(x)$.

SOLUTION Refer to the two figures shown below.

(a) The figure at the left shows a function for which $\lim_{x\to a} f(x)$ exists, but the function is not continuous at $x = a$ because the function is not defined at $x = a$.

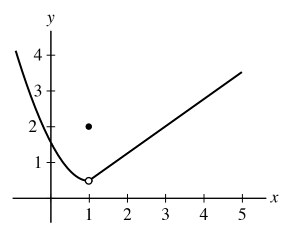
(b) The figure at the right shows a function that has a jump discontinuity at $x = a$ but $f(a)$ is not equal to either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a+} f(x)$.



In Exercises 63–66, draw the graph of a function on [0, 5] *with the given properties.*

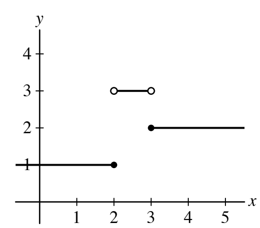
63. $f(x)$ is not continuous at $x = 1$, but $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$ exist and are equal.

SOLUTION



64. $f(x)$ is left-continuous but not continuous at $x = 2$ and right-continuous but not continuous at $x = 3$.

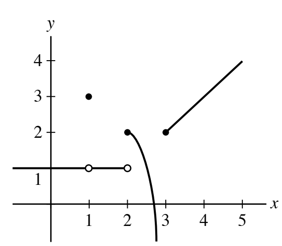
SOLUTION



65. $f(x)$ has a removable discontinuity at $x = 1$, a jump discontinuity at $x = 2$, and

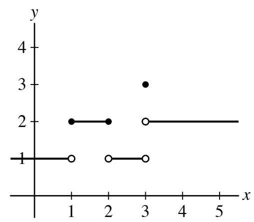
$$\lim_{x\to 3-} f(x) = -\infty, \qquad \lim_{x\to 3+} f(x) = 2$$

SOLUTION



66. $f(x)$ is right- but not left-continuous at $x = 1$, left- but not right-continuous at $x = 2$, and neither left- nor right-continuous at $x = 3$.

SOLUTION



In Exercises 67–80, evaluate using substitution.

67. $\lim_{x\to -1}(2x^3-4)$

SOLUTION $\lim_{x\to -1}(2x^3-4) = 2(-1)^3 - 4 = -6.$

68. $\lim_{x\to 2}(5x-12x^{-2})$

SOLUTION $\lim_{x\to 2}(5x-12x^{-2}) = 5(2) - 12(2^{-2}) = 10 - 12(\frac{1}{4}) = 7.$

69. $\lim_{x\to 3}\frac{x+2}{x^2+2x}$

SOLUTION $\lim_{x\to 3}\frac{x+2}{x^2+2x} = \frac{3+2}{3^2+2\cdot 3} = \frac{5}{15} = \frac{1}{3}$

70. $\lim_{x\to \pi}\sin\left(\frac{x}{2}-\pi\right)$

SOLUTION $\lim_{x\to \pi}\sin(\frac{x}{2}-\pi) = \sin(-\frac{\pi}{2}) = -1.$

71. $\lim_{x\to \frac{\pi}{4}}\tan(3x)$

SOLUTION $\lim_{x\to \frac{\pi}{4}}\tan(3x) = \tan(3\cdot\frac{\pi}{4}) = \tan(\frac{3\pi}{4}) = -1$

72. $\lim_{x\to \pi}\frac{1}{\cos x}$

SOLUTION $\lim_{x\to \pi}\frac{1}{\cos x} = \frac{1}{\cos \pi} = \frac{1}{-1} = -1.$

73. $\lim_{x\to 4}x^{-5/2}$

SOLUTION $\lim_{x\to 4}x^{-5/2} = 4^{-5/2} = \frac{1}{32}.$

74. $\lim_{x\to 2}\sqrt{x^3+4x}$

SOLUTION $\lim_{x\to 2}\sqrt{x^3+4x} = \sqrt{2^3+4(2)} = 4.$

75. $\lim_{x\to -1}(1-8x^3)^{3/2}$

SOLUTION $\lim_{x\to -1}(1-8x^3)^{3/2} = (1-8(-1)^3)^{3/2} = 27.$

76. $\lim_{x\to 2}\left(\frac{7x+2}{4-x}\right)^{2/3}$

SOLUTION $\lim_{x\to 2}\left(\frac{7x+2}{4-x}\right)^{2/3} = \left(\frac{7(2)+2}{4-2}\right)^{2/3} = 4.$

77. $\lim_{x\to 3}10^{x^2-2x}$

SOLUTION $\lim_{x\to 3}10^{x^2-2x} = 10^{3^2-2(3)} = 1000.$

78. $\lim_{x\to -\frac{\pi}{2}}3^{\sin x}$

SOLUTION $\lim_{x \to -\frac{\pi}{2}} 3^{\sin x} = 3^{\sin(-\pi/2)} = \frac{1}{3}.$

79. $\lim_{x \to 4} \sin^{-1}\left(\frac{x}{4}\right)$

SOLUTION $\lim_{x \to 4} \sin^{-1}\left(\frac{x}{4}\right) = \sin^{-1}\left(\lim_{x \to 4} \frac{x}{4}\right) = \sin^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{2}$

80. $\lim_{x \to 0} \tan^{-1}(e^x)$

SOLUTION $\lim_{x \to 0} \tan^{-1}(e^x) = \tan^{-1}\left(\lim_{x \to 0} e^x\right) = \tan^{-1}(e^0) = \tan^{-1} 1 = \frac{\pi}{4}$

81. Suppose that $f(x)$ and $g(x)$ are discontinuous at $x = c$. Does it follow that $f(x) + g(x)$ is discontinuous at $x = c$? If not, give a counterexample. Does this contradict Theorem 1 (i)?

SOLUTION Even if $f(x)$ and $g(x)$ are discontinuous at $x = c$, it is *not* necessarily true that $f(x) + g(x)$ is discontinuous at $x = c$. For example, suppose $f(x) = -x^{-1}$ and $g(x) = x^{-1}$. Both $f(x)$ and $g(x)$ are discontinuous at $x = 0$; however, the function $f(x) + g(x) = 0$, which is continuous everywhere, including $x = 0$. This does not contradict Theorem 1 (i), which deals only with continuous functions.

82. Prove that $f(x) = |x|$ is continuous for all x. *Hint:* To prove continuity at $x = 0$, consider the one-sided limits.

SOLUTION Let $c < 0$. Then

$$\lim_{x \to c} |x| = \lim_{x \to c} -x = -c = |c|.$$

Next, let $c > 0$. Then

$$\lim_{x \to c} |x| = \lim_{x \to c} x = c = |c|.$$

Finally,

$$\lim_{x \to 0-} |x| = \lim_{x \to 0-} -x = 0,$$

$$\lim_{x \to 0+} |x| = \lim_{x \to 0+} x = 0$$

and we recall that $|0| = 0$. Thus, $|x|$ is continuous for all x.

83. Use the result of Exercise 82 to prove that if $g(x)$ is continuous, then $f(x) = |g(x)|$ is also continuous.

SOLUTION Recall that the composition of two continuous functions is continuous. Now, $f(x) = |g(x)|$ is a composition of the continuous functions $g(x)$ and $|x|$, so is also continuous.

84. Which of the following quantities would be represented by continuous functions of time and which would have one or more discontinuities?

(a) Velocity of an airplane during a flight
(b) Temperature in a room under ordinary conditions
(c) Value of a bank account with interest paid yearly
(d) The salary of a teacher
(e) The population of the world

SOLUTION

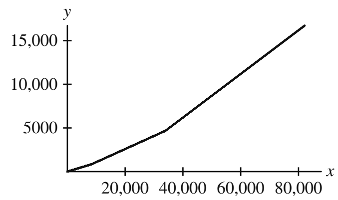
(a) The velocity of an airplane during a flight from Boston to Chicago is a continuous function of time.
(b) The temperature of a room under ordinary conditions is a continuous function of time.
(c) The value of a bank account with interest paid yearly is *not* a continuous function of time. It has discontinuities when deposits or withdrawals are made and when interest is paid.
(d) The salary of a teacher is *not* a continuous function of time. It has discontinuities whenever the teacher gets a raise (or whenever his or her salary is lowered).
(e) The population of the world is *not* a continuous function of time since it changes by a discrete amount with each birth or death. Since it takes on such large numbers (many billions), it is often treated as a continuous function for the purposes of mathematical modeling.

85. In 2009, the federal income tax $T(x)$ on income of x dollars (up to \$82,250) was determined by the formula

$$T(x) = \begin{cases} 0.10x & \text{for } 0 \le x < 8350 \\ 0.15x - 417.50 & \text{for } 8350 \le x < 33{,}950 \\ 0.25x - 3812.50 & \text{for } 33{,}950 \le x < 82{,}250 \end{cases}$$

Sketch the graph of $T(x)$. Does $T(x)$ have any discontinuities? Explain why, if $T(x)$ had a jump discontinuity, it might be advantageous in some situations to earn *less* money.

SOLUTION $T(x)$, the amount of federal income tax owed on an income of x dollars in 2009, might be a discontinuous function depending upon how the tax tables are constructed (as determined by that year's regulations). Here is a graph of $T(x)$ for that particular year.



If $T(x)$ had a jump discontinuity (say at $x = c$), it might be advantageous to earn slightly less income than c (say $c - \epsilon$) and be taxed at a lower rate than to earn c or more and be taxed at a higher rate. Your net earnings may actually be more in the former case than in the latter one.

Further Insights and Challenges

86. If $f(x)$ has a removable discontinuity at $x = c$, then it is possible to redefine $f(c)$ so that $f(x)$ is continuous at $x = c$. Can this be done in more than one way?

SOLUTION In order for $f(x)$ to have a removable discontinuity at $x = c$, $\lim_{x \to c} f(x) = L$ must exist. To remove the discontinuity, we define $f(c) = L$. Then f is continuous at $x = c$ since $\lim_{x \to c} f(x) = L = f(c)$. Now *assume* that we may define $f(c) = M \neq L$ and still have f continuous at $x = c$. Then $\lim_{x \to c} f(x) = f(c) = M$. Therefore $M = L$, a contradiction. Roughly speaking, there's only one way to fill in the hole in the graph of f!

87. Give an example of functions $f(x)$ and $g(x)$ such that $f(g(x))$ is continuous but $g(x)$ has at least one discontinuity.

SOLUTION Answers may vary. The simplest examples are the functions $f(g(x))$ where $f(x) = C$ is a constant function, and $g(x)$ is defined for all x. In these cases, $f(g(x)) = C$. For example, if $f(x) = 3$ and $g(x) = [x]$, g is discontinuous at all integer values $x = n$, but $f(g(x)) = 3$ is continuous.

88. Continuous at Only One Point Show that the following function is continuous only at $x = 0$:

$$f(x) = \begin{cases} x & \text{for } x \text{ rational} \\ -x & \text{for } x \text{ irrational} \end{cases}$$

SOLUTION Let $f(x) = x$ for x rational and $f(x) = -x$ for x irrational.

- Now $f(0) = 0$ since 0 is rational. Moreover, as $x \to 0$, we have $|f(x) - f(0)| = |f(x) - 0| = |x| \to 0$. Thus $\lim_{x \to 0} f(x) = f(0)$ and f is continuous at $x = 0$.
- Let $c \neq 0$ be any nonzero rational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of irrational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |-x_n - c| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x \to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all rational numbers.
- Let $c \neq 0$ be any nonzero irrational number. Let $\{x_1, x_2, \ldots\}$ be a sequence of rational points that approach c; i.e., as $n \to \infty$, the x_n get arbitrarily close to c. Notice that as $n \to \infty$, we have $|f(x_n) - f(c)| = |x_n - (-c)| = |x_n + c| \to |2c| \neq 0$. Therefore, it is *not* true that $\lim_{x \to c} f(x) = f(c)$. Accordingly, f is *not* continuous at $x = c$. Since c was arbitrary, f is discontinuous at all irrational numbers.
- CONCLUSION: f is continuous at $x = 0$ and is discontinuous at all points $x \neq 0$.

89. Show that $f(x)$ is a discontinuous function for all x where $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1 & \text{for } x \text{ rational} \\ -1 & \text{for } x \text{ irrational} \end{cases}$$

Show that $f(x)^2$ is continuous for all x.

SOLUTION $\lim_{x \to c} f(x)$ does not exist for any c. If c is irrational, then there is always a rational number r arbitrarily close to c so that $|f(c) - f(r)| = 2$. If, on the other hand, c is rational, there is always an *irrational* number z arbitrarily close to c so that $|f(c) - f(z)| = 2$.

On the other hand, $f(x)^2$ is a constant function that always has value 1, which is obviously continuous.