36. Let $T = \frac{3}{2}\sqrt{L}$ as in Exercise 21. The numbers in the second column of the following table are increasing, and those in the last column are decreasing. Explain why in terms of the graph of T as a function of L. Also, explain graphically why the instantaneous rate of change at $L = 3$ lies between 0.4329 and 0.4331.

Average Rates of Change of T with Respect to L

Interval	Average rate of change	Interval	Average rate of change
[3, 3.2]	0.42603	[2.8, 3]	0.44048
[3, 3.1]	0.42946	[2.9, 3]	0.43668
[3, 3.001]	0.43298	[2.999, 3]	0.43305
[3, 3.0005]	0.43299	[2.9995, 3]	0.43303

SOLUTION Since the average rate of change is increasing on the intervals $[3, L]$ as L get close to 3, we know that the slopes of the secant lines between points on the graph over these intervals are increasing. The more rows we add with smaller intervals, the greater the average rate of change. This means that the instantaneous rate of change is probably greater than all of the numbers in this column.

Likewise, since the average rate of change is *decreasing* on the intervals $[L, 3]$ as L gets closer to 3, we know that the slopes of the secant lines between points over these intervals are decreasing. This means that the instantaneous rate of change is probably less than all the numbers in this column.

The tangent slope is somewhere between the greatest value in the first column and the least value in the second column. Hence, it is between 0.43299 and 0.43303. The first column underestimates the instantaneous rate of change by secant slopes; this estimate improves as L decreases toward $L = 3$. The second column overestimates the instantaneous rate of change by secant slopes; this estimate improves as L increases toward $L = 3$.

2.2 Limits: A Numerical and Graphical Approach

Preliminary Questions

1. What is the limit of $f(x) = 1$ as $x \to \pi$?

SOLUTION $\lim_{x\to\pi} 1 = 1$.

2. What is the limit of $g(t) = t$ as $t \to \pi$?

SOLUTION $\lim_{t\to\pi} t = \pi$.

3. Is $\lim\limits_{x\to 10} 20$ equal to 10 or 20?

SOLUTION $\lim_{x\to 10} 20 = 20$.

4. Can $f(x)$ approach a limit as $x \to c$ if $f(c)$ is undefined? If so, give an example.

SOLUTION Yes. The limit of a function f as $x \to c$ does not depend on what happens *at* $x = c$, only on the behavior of f as $x \to c$. As an example, consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

The function is clearly not defined at $x = 1$ but

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} (x + 1) = 2.$$

5. What does the following table suggest about $\lim\limits_{x\to 1-} f(x)$ and $\lim\limits_{x\to 1+} f(x)$?

x	0.9	0.99	0.999	1.1	1.01	1.001
$f(x)$	7	25	4317	3.0126	3.0047	3.00011

SOLUTION The values in the table suggest that $\lim_{x\to 1-} f(x) = \infty$ and $\lim_{x\to 1+} f(x) = 3$.

6. Can you tell whether $\lim\limits_{x\to 5} f(x)$ exists from a plot of $f(x)$ for $x > 5$? Explain.

SOLUTION No. By examining values of $f(x)$ for x close to but greater than 5, we can determine whether the one-sided limit $\lim_{x\to 5+} f(x)$ exists. To determine whether $\lim_{x\to 5} f(x)$ exists, we must examine value of $f(x)$ on both sides of $x = 5$.

7. If you know in advance that $\lim_{x\to 5} f(x)$ exists, can you determine its value from a plot of $f(x)$ for all $x > 5$?

SOLUTION Yes. If $\lim_{x\to 5} f(x)$ exists, then both one-sided limits must exist and be equal.

Exercises

In Exercises 1–4, fill in the tables and guess the value of the limit.

1. $\lim_{x\to 1} f(x)$, where $f(x) = \dfrac{x^3 - 1}{x^2 - 1}$.

x	$f(x)$	x	$f(x)$
1.002		0.998	
1.001		0.999	
1.0005		0.9995	
1.00001		0.99999	

SOLUTION

x	0.998	0.999	0.9995	0.99999	1.00001	1.0005	1.001	1.002
$f(x)$	1.498501	1.499250	1.499625	1.499993	1.500008	1.500375	1.500750	1.501500

The limit as $x \to 1$ is $\frac{3}{2}$.

2. $\lim_{t\to 0} h(t)$, where $h(t) = \dfrac{\cos t - 1}{t^2}$. Note that $h(t)$ is even; that is, $h(t) = h(-t)$.

t	± 0.002	± 0.0001	± 0.00005	± 0.00001
$h(t)$				

SOLUTION

t	± 0.002	± 0.0001
$h(t)$	-0.499999833333	-0.499999999583
t	± 0.00005	± 0.00001
$h(t)$	-0.499999999896	-0.500000000000

The limit as $t \to 0$ is $-\frac{1}{2}$.

3. $\lim_{y\to 2} f(y)$, where $f(y) = \dfrac{y^2 - y - 2}{y^2 + y - 6}$.

y	$f(y)$	y	$f(y)$
2.002		1.998	
2.001		1.999	
2.0001		1.9999	

SOLUTION

y	1.998	1.999	1.9999	2.0001	2.001	2.02
$f(y)$	0.59984	0.59992	0.599992	0.600008	0.60008	0.601594

The limit as $y \to 2$ is $\frac{3}{5}$.

4. $\lim_{x\to 0+} f(x)$, where $f(x) = x \ln x$.

x	1	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$							

SOLUTION

x	1.0	0.5	0.1	0.05	0.01	0.005	0.001
$f(x)$	0	−0.34657	−0.23026	−0.14979	−0.04605	−0.02649	−0.00691

The limit as $x \to 0+$ is 0.

5. Determine $\lim_{x \to 0.5} f(x)$ for $f(x)$ as in Figure 1.

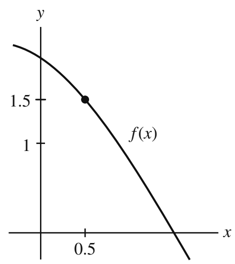


FIGURE 1

SOLUTION The graph suggests that $f(x) \to 1.5$ as $x \to 0.5$.

6. Determine $\lim_{x \to 0.5} g(x)$ for $g(x)$ as in Figure 2.

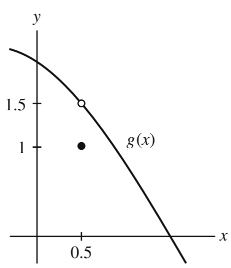


FIGURE 2

SOLUTION The graph suggests that $g(x) \to 1.5$ as $x \to 0.5$. The value $g(0.5)$, which happens to be 1, does not affect the limit.

In Exercises 7 and 8, evaluate the limit.

7. $\lim_{x \to 21} x$

SOLUTION As $x \to 21$, $f(x) = x \to 21$. You can see this, for example, on the graph of $f(x) = x$.

8. $\lim_{x \to 4.2} \sqrt{3}$

SOLUTION The graph of $f(x) = \sqrt{3}$ is a horizontal line. $f(x) = \sqrt{3}$ for all values of x, so the limit is also equal to $\sqrt{3}$.

In Exercises 9–16, verify each limit using the limit definition. For example, in Exercise 9, show that $|3x - 12|$ *can be made as small as desired by taking* x *close to* 4.

9. $\lim_{x \to 4} 3x = 12$

SOLUTION $|3x - 12| = 3|x - 4|$. $|3x - 12|$ can be made arbitrarily small by making x close enough to 4, thus making $|x - 4|$ small.

10. $\lim_{x \to 5} 3 = 3$

SOLUTION $|f(x) - 3| = |3 - 3| = 0$ for all values of x so $f(x) - 3$ is already smaller than any positive number as $x \to 5$.

11. $\lim_{x \to 3} (5x + 2) = 17$

SOLUTION $|(5x + 2) - 17| = |5x - 15| = 5|x - 3|$. Therefore, if you make $|x - 3|$ small enough, you can make $|(5x + 2) - 17|$ as small as desired.

12. $\lim_{x \to 2} (7x - 4) = 10$

SOLUTION As $x \to 2$, note that $|(7x-4)-10| = |7x-14| = 7|x-2|$. If you make $|x-2|$ small enough, you can make $|(7x-4)-10|$ as small as desired.

13. $\lim_{x\to 0} x^2 = 0$

SOLUTION As $x \to 0$, we have $|x^2 - 0| = |x+0||x-0|$. To simplify things, suppose that $|x| < 1$, so that $|x+0||x-0| = |x||x| < |x|$. By making $|x|$ sufficiently small, so that $|x+0||x-0| = x^2$ is even smaller, you can make $|x^2 - 0|$ as small as desired.

14. $\lim_{x\to 0} (3x^2 - 9) = -9$

SOLUTION $|3x^2 - 9 - (-9)| = |3x^2| = 3|x^2|$. If you make $|x| < 1$, $|x^2| < |x|$, so that making $|x-0|$ small enough can make $|3x^2 - 9 - (-9)|$ as small as desired.

15. $\lim_{x\to 0} (4x^2 + 2x + 5) = 5$

SOLUTION As $x \to 0$, we have $|4x^2 + 2x + 5 - 5| = |4x^2 + 2x| = |x||4x+2|$. If $|x| < 1$, $|4x+2|$ can be no bigger than 6, so $|x||4x+2| < 6|x|$. Therefore, by making $|x-0| = |x|$ sufficiently small, you can make $|4x^2 + 2x + 5 - 5| = |x||4x+2|$ as small as desired.

16. $\lim_{x\to 0} (x^3 + 12) = 12$

SOLUTION $|(x^3+12) - 12| = |x^3|$. If we make $|x| < 1$, then $|x^3| < |x|$. Therefore, by making $|x-0| = |x|$ sufficiently small, we can make $|(x^3+12) - 12|$ as small as desired.

In Exercises 17–36, estimate the limit numerically or state that the limit does not exist. If infinite, state whether the one-sided limits are ∞ or $-\infty$.

17. $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

SOLUTION

x	0.9995	0.99999	1.00001	1.0005
$f(x)$	0.500063	0.500001	0.49999	0.499938

The limit as $x \to 1$ is $\frac{1}{2}$.

18. $\lim_{x\to -4} \frac{2x^2 - 32}{x+4}$

SOLUTION

x	-4.001	-4.0001	-3.9999	-3.999
$f(x)$	-16.002	-16.0002	-15.9998	-15.998

The limit as $x \to -4$ is -16.

19. $\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - x - 2}$

SOLUTION

x	1.999	1.99999	2.00001	2.001
$f(x)$	1.666889	1.666669	1.666664	1.666445

The limit as $x \to 2$ is $\frac{5}{3}$.

20. $\lim_{x\to 3} \frac{x^3 - 2x^2 - 9}{x^2 - 2x - 3}$

SOLUTION

x	2.99	2.995	3.005	3.01
$f(x)$	3.741880	3.745939	3.754064	3.758130

The limit as $x \to 3$ is 3.75.

21. $\lim_{x \to 0} \frac{\sin 2x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	1.999867	1.999967	1.999967	1.999867

The limit as $x \to 0$ is 2.

22. $\lim_{x \to 0} \frac{\sin 5x}{x}$

SOLUTION

x	-0.01	-0.005	0.005	0.01
$f(x)$	4.997917	4.999479	4.999479	4.997917

The limit as $x \to 0$ is 5.

23. $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}$

SOLUTION

θ	-0.05	-0.001	0.001	0.05
$f(\theta)$	0.0249948	0.0005	-0.0005	-0.0249948

The limit as $\theta \to 0$ is 0.

24. $\lim_{x \to 0} \frac{\sin x}{x^2}$

SOLUTION

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(x)$	-99.9983	-999.9998	-10000.0	10000.0	999.9998	99.9983

The limit does not exist. As $x \to 0-$, $f(x) \to -\infty$; similarly, as $x \to 0+$, $f(x) \to \infty$.

25. $\lim_{x \to 4} \frac{1}{(x-4)^3}$

SOLUTION

x	3.99	3.999	3.9999	4.0001	4.001	4.01
$f(x)$	-10^6	-10^9	-10^{12}	10^{12}	10^9	10^6

The limit does not exist. As $x \to 4-$, $f(x) \to -\infty$; similarly, as $x \to 4+$, $f(x) \to \infty$.

26. $\lim_{x \to 1-} \frac{3-x}{x-1}$

SOLUTION

x	0.99	0.999	0.9999	0.99999
$f(x)$	-201	-2001	-20001	-200001

As $x \to 1-$, $f(x) \to -\infty$.

27. $\lim_{x \to 3+} \frac{x-4}{x^2-9}$

SOLUTION

x	3.01	3.001	3.0001	3.00001
$f(x)$	-16.473	-166.473	-1666.473	-16666.473

As $x \to 3+$, $f(x) \to -\infty$.

28. $\lim_{h\to 0} \frac{3^h - 1}{h}$

SOLUTION

h	-0.05	-0.001	-0.0001	0.0001	0.001	0.05
$f(h)$	1.06898	1.09801	1.09855	1.09867	1.09922	1.12935

The limit as $h \to 0$ is approximately 1.099. (The exact answer is $\ln 3$.)

29. $\lim_{h\to 0} \sin h \cos \frac{1}{h}$

SOLUTION

h	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(h)$	-0.008623	-0.000562	0.000095	-0.000095	0.000562	0.008623

The limit as $h \to 0$ is 0.

30. $\lim_{h\to 0} \cos \frac{1}{h}$

SOLUTION

h	± 0.1	± 0.01	± 0.001	± 0.0001
$f(h)$	-0.839072	0.862319	0.562379	-0.952155

The limit does not exist since $\cos(1/h)$ oscillates infinitely often as $h \to 0$.

31. $\lim_{x\to 0} |x|^x$

SOLUTION

x	-0.05	-0.001	-0.00001	0.00001	0.001	0.05
$f(x)$	1.161586	1.006932	1.000115	0.999885	0.993116	0.860892

The limit as $x \to 0$ is 1.

32. $\lim_{x\to 1+} \frac{\sec^{-1} x}{\sqrt{x-1}}$

SOLUTION

x	1.05	1.01	1.005	1.001
$f(x)$	1.3857	1.4084	1.4113	1.4136

The limit as $x \to 1+$ is approximately 1.414. (The exact answer is $\sqrt{2}$.)

33. $\lim_{t\to e} \frac{t - e}{\ln t - 1}$

SOLUTION

r	$e - 0.01$	$e - 0.001$	$e - 0.0001$	$e + 0.0001$	$e + 0.001$	$e + 0.01$
$f(t)$	2.713279	2.717782	2.718232	2.718332	2.718782	2.723279

The limit as $t \to e$ is approximately 2.718. (The exact answer is e.)

34. $\lim_{r\to 0} (1 + r)^{1/r}$

SOLUTION

r	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$f(r)$	2.731999	2.719642	2.718418	2.718146	2.716924	2.704814

The limit as $r \to 0$ is approximately 2.718. (The exact answer is e.)

35. $\displaystyle\lim_{x\to 1-} \frac{\tan^{-1} x}{\cos^{-1} x}$

SOLUTION

x	0.999	0.9999	0.99999	0.999999	0.9999999
$f(x)$	17.549	55.532	175.619	555.360	1756.204

The limit as $x \to 1-$ does not exist.

36. $\displaystyle\lim_{x\to 0} \frac{\tan^{-1} x - x}{\sin^{-1} x - x}$

SOLUTION

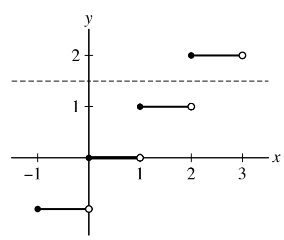
x	−0.01	−0.001	0.001	0.01
$f(x)$	−1.999791	−2.000066	−2.000066	−1.999791

The limit as $x \to 0$ is approximately -2.00. (The exact answer is -2.)

37. The **greatest integer function** is defined by $[x] = n$, where n is the unique integer such that $n \le x < n + 1$. Sketch the graph of $y = [x]$. Calculate, for c an integer:

(a) $\displaystyle\lim_{x\to c-} [x]$ **(b)** $\displaystyle\lim_{x\to c+} [x]$

SOLUTION Here is a graph of the greatest integer function:



(a) From the graph, we see that, for c an integer,

$$\lim_{x\to c-} [x] = c - 1.$$

(b) From the graph, we see that, for c an integer,

$$\lim_{x\to c+} [x] = c.$$

38. Determine the one-sided limits at $c = 1, 2$, and 4 of the function $g(x)$ shown in Figure 3, and state whether the limit exists at these points.

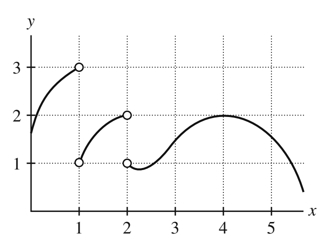


FIGURE 3

SOLUTION

- At $c = 1$, the left-hand limit is $\displaystyle\lim_{x\to 1-} g(x) = 3$, whereas the right-hand limit is $\displaystyle\lim_{x\to 1+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 1$.
- At $c = 2$, the left-hand limit is $\displaystyle\lim_{x\to 2-} g(x) = 2$, whereas the right-hand limit is $\displaystyle\lim_{x\to 2+} g(x) = 1$. Accordingly, the two-sided limit does not exist at $c = 2$.
- At $c = 4$, the left-hand limit is $\displaystyle\lim_{x\to 4-} g(x) = 2$, whereas the right-hand limit is $\displaystyle\lim_{x\to 4+} g(x) = 2$. Accordingly, the two-sided limit exists at $c = 4$ and equals 2.

In Exercises 39–46, determine the one-sided limits numerically or graphically. If infinite, state whether the one-sided limits are ∞ or $-\infty$, and describe the corresponding vertical asymptote. In Exercise 46, $[x]$ is the greatest integer function defined in Exercise 37.

39. $\lim_{x \to 0\pm} \frac{\sin x}{|x|}$

SOLUTION

x	-0.2	-0.02	0.02	0.2
$f(x)$	-0.993347	-0.999933	0.999933	0.993347

The left-hand limit is $\lim_{x \to 0-} f(x) = -1$, whereas the right-hand limit is $\lim_{x \to 0+} f(x) = 1$.

40. $\lim_{x \to 0\pm} |x|^{1/x}$

SOLUTION

x	-0.2	-0.1	0.15	0.2
$f(x)$	3125.0	10^{10}	0.000003	0.000320

The left-hand limit is $\lim_{x \to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x \to 0+} f(x) = 0$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = |x|^{1/x}$.

41. $\lim_{x \to 0\pm} \frac{x - \sin|x|}{x^3}$

SOLUTION

x	-0.1	-0.01	0.01	0.1
$f(x)$	199.853	19999.8	0.166666	0.166583

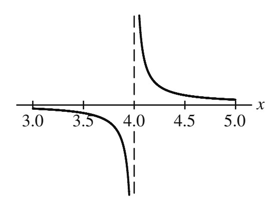
The left-hand limit is $\lim_{x \to 0-} f(x) = \infty$, whereas the right-hand limit is $\lim_{x \to 0+} f(x) = \frac{1}{6}$. Thus, the line $x = 0$ is a vertical asymptote from the left for the graph of $y = \frac{x - \sin|x|}{x^3}$.

42. $\lim_{x \to 4\pm} \frac{x + 1}{x - 4}$

SOLUTION The graph of $y = \frac{x+1}{x-4}$ for x near 4 is shown below. From this graph, we see that

$$\lim_{x \to 4-} \frac{x + 1}{x - 4} = -\infty \quad \text{while} \quad \lim_{x \to 4+} \frac{x + 1}{x - 4} = \infty.$$

Thus, the line $x = 4$ is a vertical asymptote for the graph of $y = \frac{x+1}{x-4}$.

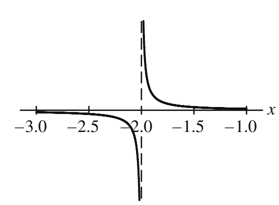


43. $\lim_{x \to -2\pm} \frac{4x^2 + 7}{x^3 + 8}$

SOLUTION The graph of $y = \frac{4x^2+7}{x^3+8}$ for x near -2 is shown below. From this graph, we see that

$$\lim_{x \to -2-} \frac{4x^2 + 7}{x^3 + 8} = -\infty \quad \text{while} \quad \lim_{x \to -2+} \frac{4x^2 + 7}{x^3 + 8} = \infty.$$

Thus, the line $x = -2$ is a vertical asymptote for the graph of $y = \frac{4x^2+7}{x^3+8}$.

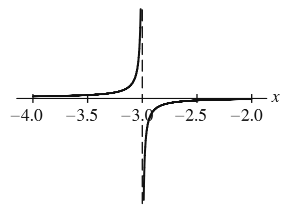


44. $\lim_{x\to -3\pm} \frac{x^2}{x^2-9}$

SOLUTION The graph of $y = \frac{x^2}{x^2-9}$ for x near -3 is shown below. From this graph, we see that

$$\lim_{x\to -3-} \frac{x^2}{x^2-9} = \infty \quad \text{while} \quad \lim_{x\to -3+} \frac{x^2}{x^2-9} = -\infty.$$

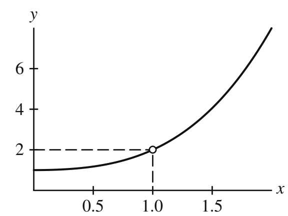
Thus, the line $x = -3$ is a vertical asymptote for the graph of $y = \frac{x^2}{x^2-9}$.



45. $\lim_{x\to 1\pm} \frac{x^5+x-2}{x^2+x-2}$

SOLUTION The graph of $y = \frac{x^5+x-2}{x^2+x-2}$ for x near 1 is shown below. From this graph, we see that

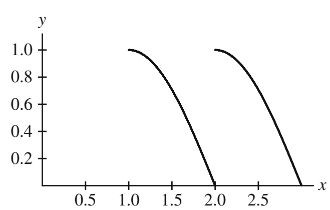
$$\lim_{x\to 1\pm} \frac{x^5+x-2}{x^2+x-2} = 2.$$



46. $\lim_{x\to 2\pm} \cos\left(\frac{\pi}{2}(x-[x])\right)$

SOLUTION The graph of $y = \cos\left(\frac{\pi}{2}(x-[x])\right)$ for x near 2 is shown below. From this graph, we see that

$$\lim_{x\to 2-} \cos\left(\frac{\pi}{2}(x-[x])\right) = 0 \quad \text{while} \quad \lim_{x\to 2+} \cos\left(\frac{\pi}{2}(x-[x])\right) = 1.$$



47. Determine the one-sided limits at $c = 2, 4$ of the function $f(x)$ in Figure 4. What are the vertical asymptotes of $f(x)$?

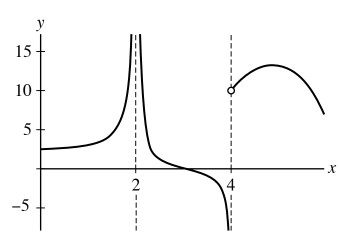


FIGURE 4

SOLUTION

- For $c = 2$, we have $\lim_{x\to 2-} f(x) = \infty$ and $\lim_{x\to 2+} f(x) = \infty$.
- For $c = 4$, we have $\lim_{x\to 4-} f(x) = -\infty$ and $\lim_{x\to 4+} f(x) = 10$.

The vertical asymptotes are the vertical lines $x = 2$ and $x = 4$.

48. Determine the infinite one- and two-sided limits in Figure 5.

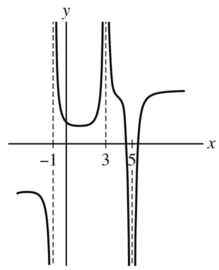


FIGURE 5

SOLUTION

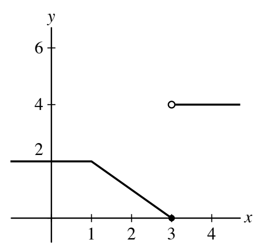
- $\lim_{x\to -1-} f(x) = -\infty$
- $\lim_{x\to -1+} f(x) = \infty$
- $\lim_{x\to 3} f(x) = \infty$
- $\lim_{x\to 5} f(x) = -\infty$

The vertical asymptotes are the vertical lines $x = 1$, $x = 3$, and $x = 5$.

In Exercises 49–52, sketch the graph of a function with the given limits.

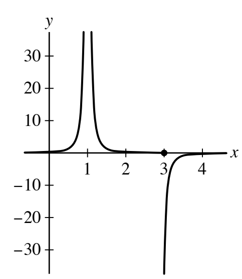
49. $\lim_{x\to 1} f(x) = 2, \quad \lim_{x\to 3-} f(x) = 0, \quad \lim_{x\to 3+} f(x) = 4$

SOLUTION



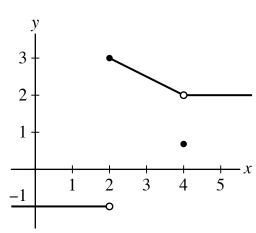
50. $\lim_{x\to 1} f(x) = \infty, \quad \lim_{x\to 3-} f(x) = 0, \quad \lim_{x\to 3+} f(x) = -\infty$

SOLUTION



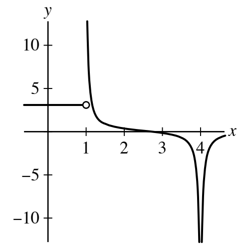
51. $\lim_{x\to 2+} f(x) = f(2) = 3, \quad \lim_{x\to 2-} f(x) = -1, \lim_{x\to 4} f(x) = 2 \neq f(4)$

SOLUTION



52. $\lim_{x\to 1+} f(x) = \infty, \quad \lim_{x\to 1-} f(x) = 3, \quad \lim_{x\to 4} f(x) = -\infty$

SOLUTION



53. Determine the one-sided limits of the function $f(x)$ in Figure 6, at the points $c = 1, 3, 5, 6$.

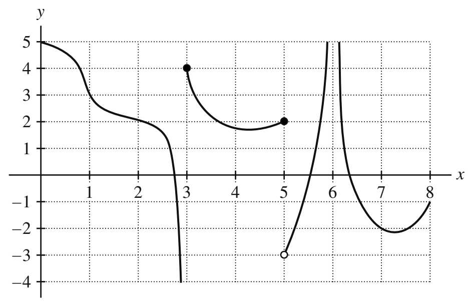
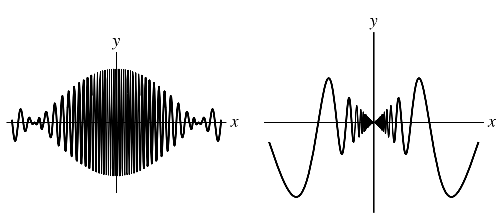


FIGURE 6 Graph of $f(x)$

SOLUTION

- $\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x) = 3$
- $\lim_{x\to 3-} f(x) = -\infty$
- $\lim_{x\to 3+} f(x) = 4$
- $\lim_{x\to 5-} f(x) = 2$
- $\lim_{x\to 5+} f(x) = -3$
- $\lim_{x\to 6-} f(x) = \lim_{x\to 6+} f(x) = \infty$

54. Does either of the two oscillating functions in Figure 7 appear to approach a limit as $x \to 0$?



(A) (B)

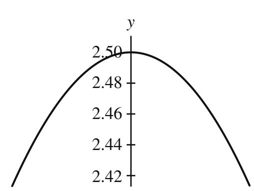
FIGURE 7

SOLUTION (A) does not appear to approach a limit as $x \to 0$; the values of the function oscillate wildly as $x \to 0$. The values of the function graphed in (B) seem to settle to 0 as $x \to 0$, so the limit seems to exist.

GU *In Exercises 55–60, plot the function and use the graph to estimate the value of the limit.*

55. $\lim_{\theta\to 0} \dfrac{\sin 5\theta}{\sin 2\theta}$

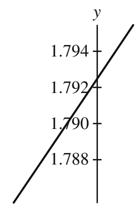
SOLUTION



From the graph of $y = \dfrac{\sin 5\theta}{\sin 2\theta}$ shown above, we see that the limit as $\theta \to 0$ is $\frac{5}{2}$.

56. $\displaystyle\lim_{x\to 0} \frac{12^x - 1}{4^x - 1}$

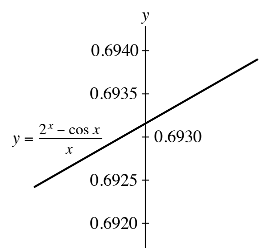
SOLUTION



From the graph of $y = \dfrac{12^x - 1}{4^x - 1}$ shown above, we see that the limit as $x \to 0$ is approximately 1.7925. (The exact answer is $\ln 12/\ln 4$.)

57. $\displaystyle\lim_{x\to 0} \frac{2^x - \cos x}{x}$

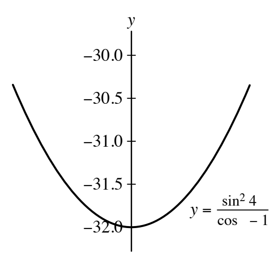
SOLUTION



The limit as $x \to 0$ is approximately 0.693. (The exact answer is $\ln 2$.)

58. $\displaystyle\lim_{\theta\to 0} \frac{\sin^2 4\theta}{\cos\theta - 1}$

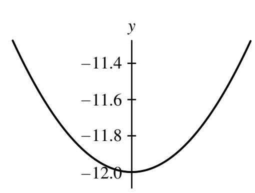
SOLUTION



The limit as $\theta \to 0$ is -32.

59. $\displaystyle\lim_{\theta\to 0} \frac{\cos 7\theta - \cos 5\theta}{\theta^2}$

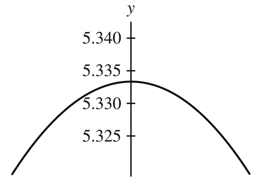
SOLUTION



From the graph of $y = \dfrac{\cos 7\theta - \cos 5\theta}{\theta^2}$ shown above, we see that the limit as $\theta \to 0$ is -12.

60. $\displaystyle\lim_{\theta\to 0} \frac{\sin^2 2\theta - \theta\sin 4\theta}{\theta^4}$

SOLUTION



From the graph of $y = \dfrac{\sin^2 2\theta - \theta \sin 4\theta}{\theta^4}$ shown above, we see that the limit as $\theta \to 0$ is approximately 5.333. (The exact answer is $\frac{16}{3}$.)

61. Let n be a positive integer. For which n are the two infinite one-sided limits $\lim_{x\to 0\pm} 1/x^n$ equal?

SOLUTION First, suppose that n is even. Then $x^n \geq 0$ for all x, and $\frac{1}{x^n} > 0$ for all $x \neq 0$. Hence,

$$\lim_{x\to 0-} \frac{1}{x^n} = \lim_{x\to 0+} \frac{1}{x^n} = \infty.$$

Next, suppose that n is odd. Then $\frac{1}{x^n} > 0$ for all $x > 0$ but $\frac{1}{x^n} < 0$ for all $x < 0$. Thus,

$$\lim_{x\to 0-} \frac{1}{x^n} = -\infty \quad \text{but} \quad \lim_{x\to 0+} \frac{1}{x^n} = \infty.$$

Finally, the two infinite one-sided limits are equal whenever n is even.

62. Let $L(n) = \lim_{x\to 1} \left(\dfrac{n}{1-x^n} - \dfrac{1}{1-x} \right)$ for n a positive integer. Investigate $L(n)$ numerically for several values of n, and then guess the value of of $L(n)$ in general.

SOLUTION

- We first notice that for $n = 1$,

$$\frac{1}{1-x} - \frac{1}{1-x} = 0,$$

so $L(1) = 0$.

- Next, let's try $n = 3$. From the table below, it appears that $L(3) = 1$.

x	0.99	0.999	1.001	1.01
$f(x)$	1.006700	1.000667	0.999334	0.993367

- For $n = 6$, we find

x	0.99	0.999	0.9999	1.0001	1.001	1.01
$f(x)$	2.529312	2.502919	2.500392	2.499375	2.497082	2.470980

Thus, $L(6) = 2.5 = \frac{5}{2}$

From these values, we conjecture that $L(n) = \frac{n-1}{2}$.

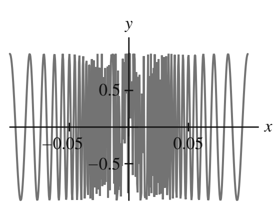
63. GU In some cases, numerical investigations can be misleading. Plot $f(x) = \cos \frac{\pi}{x}$.

(a) Does $\lim_{x\to 0} f(x)$ exist?

(b) Show, by evaluating $f(x)$ at $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$, that you might be able to trick your friends into believing that the limit exists and is equal to $L = 1$.

(c) Which sequence of evaluations might trick them into believing that the limit is $L = -1$.

SOLUTION Here is the graph of $f(x)$.



(a) From the graph of $f(x)$, which shows that the value of $f(x)$ oscillates more and more rapidly as $x \to 0$, it follows that $\lim_{x\to 0} f(x)$ does not exist.

(b) Notice that

$$f\left(\pm\frac{1}{2}\right) = \cos \pm\frac{\pi}{1/2} = \cos \pm 2\pi = 1;$$

$$f\left(\pm\frac{1}{4}\right) = \cos \pm\frac{\pi}{1/4} = \cos \pm 4\pi = 1;$$

$$f\left(\pm\frac{1}{6}\right) = \cos \pm\frac{\pi}{1/6} = \cos \pm 6\pi = 1;$$

and, in general, $f(\pm\frac{1}{2n}) = 1$ for all integers n.

(c) At $x = \pm 1, \pm\frac{1}{3}, \pm\frac{1}{5}, \ldots$, the value of $f(x)$ is always -1.

Further Insights and Challenges

64. Light waves of frequency λ passing through a slit of width a produce a **Fraunhofer diffraction pattern** of light and dark fringes (Figure 8). The intensity as a function of the angle θ is

$$I(\theta) = I_m\left(\frac{\sin(R\sin\theta)}{R\sin\theta}\right)^2$$

where $R = \pi a/\lambda$ and I_m is a constant. Show that the intensity function is not defined at $\theta = 0$. Then choose any two values for R and check numerically that $I(\theta)$ approaches I_m as $\theta \to 0$.

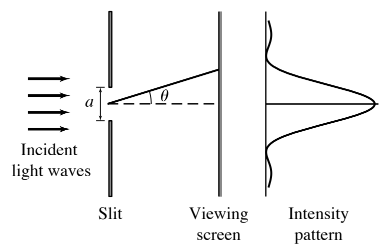


FIGURE 8 Fraunhofer diffraction pattern.

SOLUTION If you plug in $\theta = 0$, you get a division by zero in the expression

$$\frac{\sin\left(R\sin\theta\right)}{R\sin\theta};$$

thus, $I(0)$ is undefined. If $R = 2$, a table of values as $\theta \to 0$ follows:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.998667\ I_m$	$0.9999667\ I_m$	$0.9999667\ I_m$	$0.9998667\ I_m$

The limit as $\theta \to 0$ is $1 \cdot I_m = I_m$.

If $R = 3$, the table becomes:

θ	-0.01	-0.005	0.005	0.01
$I(\theta)$	$0.999700\ I_m$	$0.999925\ I_m$	$0.999925\ I_m$	$0.999700\ I_m$

Again, the limit as $\theta \to 0$ is $1 I_m = I_m$.

65. Investigate $\lim_{\theta\to 0} \frac{\sin n\theta}{\theta}$ numerically for several values of n. Then guess the value in general.

SOLUTION

- For $n = 3$, we have

θ	-0.1	-0.01	-0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	2.955202	2.999550	2.999996	2.999996	2.999550	2.955202

The limit as $\theta \to 0$ is 3.

- For $n = -5$, we have

θ	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{\sin n\theta}{\theta}$	−4.794255	−4.997917	−4.999979	−4.999979	−4.997917	−4.794255

The limit as $\theta \to 0$ is -5.

- We surmise that, in general, $\lim_{\theta \to 0} \frac{\sin n\theta}{\theta} = n$.

66. Show numerically that $\lim_{x \to 0} \frac{b^x - 1}{x}$ for $b = 3, 5$ appears to equal $\ln 3$, $\ln 5$, where $\ln x$ is the natural logarithm. Then make a conjecture (guess) for the value in general and test your conjecture for two additional values of b.

SOLUTION

-

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{5^x - 1}{x}$	1.486601	1.596556	1.608144	1.610734	1.622459	1.746189

We have $\ln 5 \approx 1.6094$.

-

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{3^x - 1}{x}$	1.040415	1.092600	1.098009	1.099216	1.104669	1.161232

We have $\ln 3 \approx 1.0986$.

- We conjecture that $\lim_{x \to 0} \frac{b^x - 1}{x} = \ln b$ for any positive number b. Here are two additional test cases.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{(\frac{1}{2})^x - 1}{x}$	−0.717735	−0.695555	−0.693387	−0.692907	−0.690750	−0.669670

We have $\ln \frac{1}{2} \approx -0.69315$.

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$\frac{7^x - 1}{x}$	1.768287	1.927100	1.944018	1.947805	1.964966	2.148140

We have $\ln 7 \approx 1.9459$.

67. Investigate $\lim_{x \to 1} \frac{x^n - 1}{x^m - 1}$ for (m, n) equal to $(2, 1)$, $(1, 2)$, $(2, 3)$, and $(3, 2)$. Then guess the value of the limit in general and check your guess for two additional pairs.

SOLUTION

-

x	0.99	0.9999	1.0001	1.01
$\frac{x - 1}{x^2 - 1}$	0.502513	0.500025	0.499975	0.497512

The limit as $x \to 1$ is $\frac{1}{2}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^2 - 1}{x - 1}$	1.99	1.9999	2.0001	2.01

The limit as $x \to 1$ is 2.

x	0.99	0.9999	1.0001	1.01
$\frac{x^2-1}{x^3-1}$	0.670011	0.666700	0.666633	0.663344

The limit as $x \to 1$ is $\frac{2}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^2-1}$	1.492513	1.499925	1.500075	1.507512

The limit as $x \to 1$ is $\frac{3}{2}$.

- For general m and n, we have $\lim_{x\to 1} \frac{x^n-1}{x^m-1} = \frac{n}{m}$.
-

x	0.99	0.9999	1.0001	1.01
$\frac{x-1}{x^3-1}$	0.336689	0.333367	0.333300	0.330022

The limit as $x \to 1$ is $\frac{1}{3}$.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x-1}$	2.9701	2.9997	3.0003	3.0301

The limit as $x \to 1$ is 3.

x	0.99	0.9999	1.0001	1.01
$\frac{x^3-1}{x^7-1}$	0.437200	0.428657	0.428486	0.420058

The limit as $x \to 1$ is $\frac{3}{7} \approx 0.428571$.

68. Find by numerical experimentation the positive integers k such that $\lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^k}$ exists.

SOLUTION

- For $k = 1$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x} = 0$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-0.01	-0.0001	0.0001	0.01

- For $k = 2$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^2} = 1$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	0.999967	1.000000	1.000000	0.999967

- For $k = 3$, the limit does not exist.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-10^2	-10^4	10^4	10^2

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^3} \to \infty$.

- For $k = 4$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^4} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^4	10^8	10^8	10^4

- For $k = 5$, the limit does not exist.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	-10^6	-10^{12}	10^{12}	10^6

Indeed, as $x \to 0-$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to -\infty$, whereas as $x \to 0+$, $f(x) = \frac{\sin(\sin^2 x)}{x^5} \to \infty$.

- For $k = 6$, we have $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin(\sin^2 x)}{x^6} = \infty$.

x	-0.01	-0.0001	0.0001	0.01
$f(x)$	10^8	10^{16}	10^{16}	10^8

- SUMMARY
  - For $k = 1$, the limit is 0.
  - For $k = 2$, the limit is 1.
  - For odd $k > 2$, the limit does not exist.
  - For even $k > 2$, the limit is ∞.

69. GU Plot the graph of $f(x) = \frac{2^x - 8}{x - 3}$.

(a) Zoom in on the graph to estimate $L = \lim_{x\to 3} f(x)$.

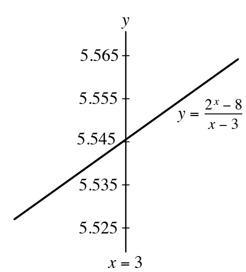
(b) Explain why

$$f(2.99999) \le L \le f(3.00001)$$

Use this to determine L to three decimal places.

SOLUTION

(a)



(b) It is clear that the graph of f rises as we move to the right. Mathematically, we may express this observation as: whenever $u < v$, $f(u) < f(v)$. Because

$$2.99999 < 3 = \lim_{x\to 3} f(x) < 3.00001,$$

it follows that

$$f(2.99999) < L = \lim_{x\to 3} f(x) < f(3.00001).$$

With $f(2.99999) \approx 5.54516$ and $f(3.00001) \approx 5.545195$, the above inequality becomes $5.54516 < L < 5.545195$; hence, to three decimal places, $L = 5.545$.

70. GU The function $f(x) = \frac{2^{1/x} - 2^{-1/x}}{2^{1/x} + 2^{-1/x}}$ is defined for $x \ne 0$.

(a) Investigate $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$ numerically.

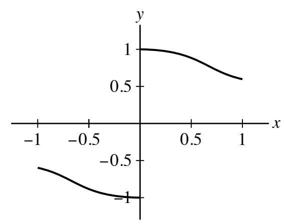
(b) Plot the graph of f and describe its behavior near $x = 0$.

SOLUTION

(a)

x	-0.3	-0.2	-0.1	0.1	0.2	0.3
$f(x)$	-0.980506	-0.998049	-0.999998	0.999998	0.998049	0.980506

(b) As $x \to 0-$, $f(x) \to -1$, whereas as $x \to 0+$, $f(x) \to 1$.



2.3 Basic Limit Laws

Preliminary Questions

1. State the Sum Law and Quotient Law.

SOLUTION Suppose $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist. The Sum Law states that

$$\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

Provided $\lim_{x\to c} g(x) \neq 0$, the Quotient Law states that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

2. Which of the following is a verbal version of the Product Law (assuming the limits exist)?

(a) The product of two functions has a limit.

(b) The limit of the product is the product of the limits.

(c) The product of a limit is a product of functions.

(d) A limit produces a product of functions.

SOLUTION The verbal version of the Product Law is **(b)**: The limit of the product is the product of the limits.

3. Which statement is correct? The Quotient Law does not hold if:

(a) The limit of the denominator is zero.

(b) The limit of the numerator is zero.

SOLUTION Statement **(a)** is correct. The Quotient Law does not hold if the limit of the denominator is zero.

Exercises

In Exercises 1–24, evaluate the limit using the Basic Limit Laws and the limits $\lim_{x\to c} x^{p/q} = c^{p/q}$ *and* $\lim_{x\to c} k = k$.

1. $\lim_{x\to 9} x$

SOLUTION $\lim_{x\to 9} x = 9$.

2. $\lim_{x\to -3} 14$

SOLUTION $\lim_{x\to -3} 14 = 14$.

3. $\lim_{x\to \frac{1}{2}} x^4$

SOLUTION $\lim_{x\to \frac{1}{2}} x^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.