SOLUTION  If \( n \neq -1 \), then
\[
\frac{d}{dx} F(x) = \frac{d}{dx} \left( \frac{x^{n+1} - 1}{n+1} \right) = x^n.
\]

Therefore, \( F(x) \) is an antiderivative of \( y = x^n \). Using L'Hôpital’s Rule,
\[
\lim_{n \to 1} F(x) = \lim_{n \to 1} \frac{x^{n+1} - 1}{n+1} = \lim_{n \to 1} \frac{x^{n+1} \ln x}{1} = \ln x.
\]

### CHAPTER REVIEW EXERCISES

In Exercises 1–6, estimate using the Linear Approximation or linearization, and use a calculator to estimate the error.

1. \( 8.1^{1/3} - 2 \)

**SOLUTION**  Let \( f(x) = x^{1/3}, a = 8 \) and \( \Delta x = 0.1 \). Then \( f'(x) = \frac{1}{3}x^{-2/3}, \ f'(a) = \frac{1}{12} \) and, by the Linear Approximation,
\[
\Delta f = 8.1^{1/3} - 2 \approx f'(a)\Delta x = \frac{1}{12} (0.1) = 0.00833333.
\]

Using a calculator, \( 8.1^{1/3} - 2 = 0.00829885 \). The error in the Linear Approximation is therefore
\[
|0.00829885 - 0.00833333| = 3.445 \times 10^{-5}.
\]

2. \( \frac{1}{\sqrt{4.1}} - \frac{1}{2} \)

**SOLUTION**  Let \( f(x) = x^{-1/2}, a = 4 \) and \( \Delta x = 0.1 \). Then \( f'(x) = -\frac{1}{2}x^{-3/2}, \ f'(a) = -\frac{1}{16} \) and, by the Linear Approximation,
\[
\Delta f = \frac{1}{\sqrt{4.1}} - \frac{1}{2} \approx f'(a)\Delta x = -\frac{1}{16} (0.1) = -0.00625.
\]

Using a calculator,
\[
\frac{1}{\sqrt{4.1}} - \frac{1}{2} = -0.00613520.
\]

The error in the Linear Approximation is therefore
\[
| -0.00613520 - (-0.00625)| = 1.148 \times 10^{-4}.
\]

3. \( 625^{1/4} - 624^{1/4} \)

**SOLUTION**  Let \( f(x) = x^{1/4}, a = 625 \) and \( \Delta x = -1 \). Then \( f'(x) = \frac{1}{4}x^{-3/4}, \ f'(a) = \frac{1}{500} \) and, by the Linear Approximation,
\[
\Delta f = 624^{1/4} - 625^{1/4} \approx f'(a)\Delta x = \frac{1}{500} (-1) = -0.002.
\]

Thus \( 625^{1/4} - 624^{1/4} \approx 0.002 \). Using a calculator,
\[
625^{1/4} - 624^{1/4} = 0.00200120.
\]

The error in the Linear Approximation is therefore
\[
|0.00200120 - (0.002)| = 1.201 \times 10^{-6}.
\]

4. \( \sqrt{101} \)

**SOLUTION**  Let \( f(x) = \sqrt{x} \) and \( a = 100 \). Then \( f(a) = 10, \ f'(x) = \frac{1}{2}x^{-1/2} \) and \( f'(a) = \frac{1}{20} \). The linearization of \( f(x) \) at \( a = 100 \) is therefore
\[
L(x) = f(a) + f'(a)(x - a) = 10 + \frac{1}{20} (x - 100),
\]
and \( \sqrt{101} \approx L(101) = 10.05 \). Using a calculator, \( \sqrt{101} = 10.049876 \), so the error in the Linear Approximation is
\[
|10.049876 - 10.05| = 1.244 \times 10^{-4}.
\]
5. \( \frac{1}{1.02} \)

**SOLUTION** Let \( f(x) = x^{-1} \) and \( a = 1 \). Then \( f(a) = 1 \), \( f'(x) = -x^{-2} \) and \( f'(a) = -1 \). The linearization of \( f(x) \) at \( a = 1 \) is therefore

\[
L(x) = f(a) + f'(a)(x - a) = 1 - (x - 1) = 2 - x.
\]

and \( \frac{1}{1.02} \approx L(1.02) = 0.98 \). Using a calculator, \( \frac{1}{1.02} = 0.980392 \), so the error in the Linear Approximation is

\[
|0.980392 - 0.98| = 3.922 \times 10^{-4}.
\]

6. \( \sqrt[3]{33} \)

**SOLUTION** Let \( f(x) = x^{1/5} \) and \( a = 32 \). Then \( f(a) = 2 \), \( f'(x) = \frac{1}{5}x^{-4/5} \) and \( f'(a) = \frac{1}{80} \). The linearization of \( f(x) \) at \( a = 32 \) is therefore

\[
L(x) = f(a) + f'(a)(x - a) = 2 + \frac{1}{80}(x - 32),
\]

and \( \sqrt[3]{33} \approx L(33) = 2.0125 \). Using a calculator, \( \sqrt[3]{33} = 2.012347 \), so the error in the Linear Approximation is

\[
|2.012347 - 2.0125| = 1.534 \times 10^{-4}.
\]

**In Exercises 7–12, find the linearization at the point indicated.**

7. \( y = \sqrt[3]{x} \), \( a = 25 \)

**SOLUTION** Let \( y = \sqrt[3]{x} \) and \( a = 25 \). Then \( y(a) = 5 \), \( y' = \frac{1}{3}x^{-2/3} \) and \( y'(a) = \frac{1}{450} \). The linearization of \( y \) at \( a = 25 \) is therefore

\[
L(x) = y(a) + y'(a)(x - 25) = 5 + \frac{1}{450}(x - 25).
\]

8. \( v(t) = 32t - 4t^2 \)，\( a = 2 \)

**SOLUTION** Let \( v(t) = 32t - 4t^2 \) and \( a = 2 \). Then \( v(a) = 48 \), \( v'(t) = 32 - 8t \) and \( v'(a) = 16 \). The linearization of \( v(t) \) at \( a = 2 \) is therefore

\[
L(t) = v(a) + v'(a)(t - a) = 48 + 16(t - 2) = 16t + 16.
\]

9. \( A(r) = \frac{4}{3}\pi r^3 \)，\( a = 3 \)

**SOLUTION** Let \( A(r) = \frac{4}{3}\pi r^3 \) and \( a = 3 \). Then \( A(a) = 36\pi \), \( A'(r) = 4\pi r^2 \) and \( A'(a) = 36\pi \). The linearization of \( A(r) \) at \( a = 3 \) is therefore

\[
L(r) = A(a) + A'(a)(r - a) = 36\pi + 36\pi(r - 3) = 36\pi(r - 2).
\]

10. \( V(h) = 4h(2-h)(4-2h) \)，\( a = 1 \)

**SOLUTION** Let \( V(h) = 4h(2-h)(4-2h) = 32h - 32h^2 + 8h^3 \) and \( a = 1 \). Then \( V(a) = 8 \), \( V'(h) = 32 - 64h + 24h^2 \) and \( V'(a) = -8 \). The linearization of \( V(h) \) at \( a = 1 \) is therefore

\[
L(h) = V(a) + V'(a)(h - a) = 8 - 8(h - 1) = 16 - 8h.
\]

11. \( P(x) = e^{-x^2/2} \)，\( a = 1 \)

**SOLUTION** Let \( P(x) = e^{-x^2/2} \) and \( a = 1 \). Then \( P(a) = e^{-1/2} \), \( P'(x) = -xe^{-x^2/2} \), and \( P'(a) = -e^{-1/2} \). The linearization of \( P(x) \) at \( a = 1 \) is therefore

\[
L(x) = P(a) + P'(a)(x - a) = e^{-1/2} - e^{-1/2}(x - 1) = \frac{1}{\sqrt{e}}(2 - x).
\]

12. \( f(x) = \ln(x + e) \)，\( a = e \)

**SOLUTION** Let \( f(x) = \ln(x + e) \) and \( a = e \). Then \( f(a) = \ln(2e) = 1 + \ln 2 \), \( f'(x) = \frac{1}{x + e} \), and \( f'(a) = \frac{1}{2e} \). The linearization of \( f(x) \) at \( a = e \) is therefore

\[
L(x) = f(a) + f'(a)(x - a) = 1 + \ln 2 + \frac{1}{2e}(x - e).
\]
In Exercises 13–18, use the Linear Approximation.

13. The position of an object in linear motion at time $t$ is $s(t) = 0.4t^2 + (t + 1)^{-1}$. Estimate the distance traveled over the time interval $[4, 4.2]$.

**SOLUTION** Let $s(t) = 0.4t^2 + (t + 1)^{-1}$, $a = 4$ and $\Delta t = 0.2$. Then $s'(t) = 0.8t - (t + 1)^{-2}$ and $s'(4) = 3.16$. Using the Linear Approximation, the distance traveled over the time interval $[4, 4.2]$ is approximately

$$
\Delta s = s(4.2) - s(4) \approx s'(4)\Delta t = 3.16(0.2) = 0.632.
$$

14. A bond that pays $10,000 in 6 years is offered for sale at a price $P$. The percentage yield $Y$ of the bond is

$$
Y = 100 \left( \frac{10,000}{P} \right)^{1/6} - 1
$$

Verify that if $P = 7500$, then $Y = 4.91\%$. Estimate the drop in yield if the price rises to $7700.$

**SOLUTION** Let $P = 7500$. Then

$$
Y = 100 \left( \frac{10,000}{7500} \right)^{1/6} - 1 = 4.91\%.
$$

If the price is raised to $7700$, then $\Delta P = 200$. With

$$
dY = \frac{dY}{dP} \Delta P = -\frac{1}{6} 100(10,000)^{1/6} p^{-7/6} = -\frac{10^{8/3}}{6} p^{-7/6},
$$

we estimate using the Linear Approximation that

$$
\Delta Y \approx Y'(7500) \Delta P = -0.46\%.
$$

15. When a bus pass from Albuquerque to Los Alamos is priced at $p$ dollars, a bus company takes in a monthly revenue of $R(p) = 1.5p - 0.01p^2$ (in thousands of dollars).

(a) Estimate $\Delta R$ if the price rises from $50$ to $53$.

(b) If $p = 80$, how will revenue be affected by a small increase in price? Explain using the Linear Approximation.

**SOLUTION**

(a) If the price is raised from $50$ to $53$, then $\Delta p = 3$ and

$$
\Delta R \approx R'(50) \Delta p = (1.5 - 0.02(50))(3) = 1.5
$$

We therefore estimate an increase of $\$1500$ in revenue.

(b) Because $R'(80) = 1.5 - 0.02(80) = -0.1$, the Linear Approximation gives $\Delta R \approx -0.1 \Delta p$. A small increase in price would thus result in a decrease in revenue.

16. A store sells 80 MP4 players per week when the players are priced at $P = 75$. Estimate the number $N$ sold if $P$ is raised to $80$, assuming that $dN/dP = -4$. Estimate $N$ if the price is lowered to $69$.

**SOLUTION** If $P$ is raised to $80$, then $\Delta P = 5$. With the assumption that $dN/dP = -4$, we estimate, using the Linear Approximation, that

$$
\Delta N \approx \frac{dN}{dP} \Delta P = (-4)(5) = -20;
$$

therefore, we estimate that only 60 MP4 players will be sold per week when the price is $80$. On the other hand, if the price is lowered to $69$, then $\Delta P = -6$ and $\Delta N \approx (-4)(-6) = 24$. We therefore estimate that 104 MP4 players will be sold per week when the price is $69$.

17. The circumference of a sphere is measured at $C = 100$ cm. Estimate the maximum percentage error in $V$ if the error in $C$ is at most 3 cm.

**SOLUTION** The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the circumference is $C = 2\pi r$, where $r$ is the radius of the sphere. Thus, $r = \frac{C}{2\pi}$ and

$$
V = \frac{4}{3} \pi \left( \frac{C}{2\pi} \right)^3 = \frac{1}{6\pi^2} C^3.
$$

Using the Linear Approximation,

$$
\Delta V \approx \frac{dV}{dC} \Delta C = \frac{1}{2\pi^2} C^2 \Delta C.
$$
so
\[
\frac{\Delta V}{V} \approx \frac{1}{2\pi^2 C^2 \Delta C} = \frac{3\Delta C}{C}.
\]

With \( C = 100 \text{ cm} \) and \( \Delta C \) at most 3 cm, we estimate that the maximum percentage error in \( V \) is \( 3 \frac{\Delta C}{C} = 0.09 \), or 9%.

18. Show that \( \sqrt{a^2 + b} \approx a + \frac{b}{2a} \) if \( b \) is small. Use this to estimate \( \sqrt{26} \) and find the error using a calculator.

**SOLUTION** Let \( a > 0 \) and let \( f(b) = \sqrt{a^2 + b} \). Then

\[
f'(b) = \frac{1}{2\sqrt{a^2 + b}}.
\]

By the Linear Approximation, \( f(b) \approx f(0) + f'(0)b \), so

\[
\sqrt{a^2 + b} \approx a + \frac{b}{2a}.
\]

To estimate \( \sqrt{26} \), let \( a = 5 \) and \( b = 1 \). Then

\[
\sqrt{26} = \sqrt{5^2 + 1} \approx 5 + \frac{1}{10} = 5.1.
\]

The error in this estimate is \(|\sqrt{26} - 5.1| = 9.80 \times 10^{-4} \).

19. Use the Intermediate Value Theorem to prove that \( \sin x - \cos x = 3x \) has a solution, and use Rolle’s Theorem to show that this solution is unique.

**SOLUTION** Let \( f(x) = \sin x - \cos x - 3x \), and observe that each root of this function corresponds to a solution of the equation \( \sin x - \cos x = 3x \). Now,

\[
f\left(\frac{\pi}{2}\right) = -1 + \frac{3\pi}{2} > 0 \quad \text{and} \quad f(0) = -1 < 0.
\]

Because \( f \) is continuous on \((-\frac{\pi}{2}, 0)\) and \( f\left(-\frac{\pi}{2}\right) \) and \( f(0) \) are of opposite sign, the Intermediate Value Theorem guarantees there exists \( c \in \left(-\frac{\pi}{2}, 0\right) \) such that \( f(c) = 0 \). Thus, the equation \( \sin x - \cos x = 3x \) has at least one solution.

Next, suppose that the equation \( \sin x - \cos x = 3x \) has two solutions, and therefore \( f(x) \) has two roots, say \( a \) and \( b \). Because \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) = 0 \), Rolle’s Theorem guarantees there exists \( c \in (a, b) \) such that \( f'(c) = 0 \). However,

\[
f'(x) = \cos x + \sin x - 3 \leq -1
\]

for all \( x \). We have reached a contradiction. Consequently, \( f(x) \) has a unique root and the equation \( \sin x - \cos x = 3x \) has a unique solution.

20. Show that \( f(x) = 2x^3 + 2x + \sin x + 1 \) has precisely one real root.

**SOLUTION** We have \( f(0) = 1 \) and \( f(-1) = -3 + \sin(-1) = -3.84 < 0 \). Therefore \( f(x) \) has a root in the interval \([-1, 0]\).

Now, suppose that \( f(x) \) has two real roots, say \( a \) and \( b \). Because \( f(x) \) is continuous on \([a, b]\) and differentiable on \((a, b)\) and \( f(a) = f(b) = 0 \), Rolle’s Theorem guarantees that there exists \( c \in (a, b) \) such that \( f'(c) = 0 \). However

\[
f'(x) = 6x^2 + 2 + \cos x > 0
\]

for all \( x \) (since \( 2 + \cos x \geq 0 \)). We have reached a contradiction. Consequently, \( f(x) \) must have precisely one real root.

21. Verify the MVT for \( f(x) = \ln x \) on \([1, 4]\).

**SOLUTION** Let \( f(x) = \ln x \). On the interval \([1, 4]\), this function is continuous and differentiable, so the MVT applies. Now, \( f'(x) = \frac{1}{x} \), so

\[
\frac{1}{c} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln 4 - \ln 1}{4 - 1} = \frac{1}{3} \ln 4.
\]

or

\[
c = \frac{3}{\ln 4} \approx 2.164 \in (1, 4).
\]

22. Suppose that \( f(1) = 5 \) and \( f'(x) \geq 2 \) for \( x \geq 1 \). Use the MVT to show that \( f(8) \geq 19 \).

**SOLUTION** Because \( f \) is continuous on \([1, 8]\) and differentiable on \((1, 8)\), the Mean Value Theorem guarantees there exists a \( c \in (1, 8) \) such that

\[
f'(c) = \frac{f(8) - f(1)}{8 - 1} \quad \text{or} \quad f(8) = f(1) + 7f'(c).
\]

Now, we are given that \( f(1) = 5 \) and that \( f'(x) \geq 2 \) for \( x \geq 1 \). Therefore,

\[
f(8) \geq 5 + 7(2) = 19.
\]
23. Use the MVT to prove that if \( f'(x) \leq 2 \) for \( x > 0 \) and \( f(0) = 4 \), then \( f(x) \leq 2x + 4 \) for all \( x \geq 0 \).

**SOLUTION**  Let \( x > 0 \). Because \( f \) is continuous on \([0, x]\) and differentiable on \((0, x)\), the Mean Value Theorem guarantees there exists a \( c \in (0, x) \) such that

\[
f'(c) = \frac{f(x) - f(0)}{x - 0} \quad \text{or} \quad f(x) = f(0) + xf'(c).
\]

Now, we are given that \( f(0) = 4 \) and that \( f'(x) \leq 2 \) for \( x > 0 \). Therefore, for all \( x \geq 0 \),

\[
f(x) \leq 4 + x(2) = 2x + 4.
\]

24. A function \( f(x) \) has derivative \( f'(x) = \frac{1}{x^4 + 1} \). Where on the interval \([1,4]\) does \( f(x) \) take on its maximum value?

**SOLUTION**  Let

\[
f'(x) = \frac{1}{x^4 + 1}.
\]

Because \( f'(x) \) is never 0 and exists for all \( x \), the function \( f \) has no critical points on the interval \([1,4]\) and so must take its maximum value at one of the interval endpoints. Moreover, as \( f'(x) \) > 0 for all \( x \), the function \( f \) is increasing for all \( x \). Consequently, on the interval \([1,4]\), the function \( f \) must take its maximum value at \( x = 4 \).

**In Exercises 25–30, find the critical points and determine whether they are minima, maxima, or neither.**

25. \( f(x) = x^3 - 4x^2 + 4x \)

**SOLUTION**  Let \( f(x) = x^3 - 4x^2 + 4x \). Then \( f'(x) = 3x^2 - 8x + 4 = (3x - 2)(x - 2) \), so that \( x = \frac{2}{3} \) and \( x = 2 \) are critical points. Next, \( f''(x) = 6x - 8 \), so \( f''(\frac{2}{3}) = -4 < 0 \) and \( f''(2) = 4 > 0 \). Therefore, by the Second Derivative Test, \( f(\frac{2}{3}) \) is a local maximum while \( f(2) \) is a local minimum.

26. \( s(t) = t^4 - 8t^2 \)

**SOLUTION**  Let \( s(t) = t^4 - 8t^2 \). Then \( s'(t) = 4t^3 - 16t = 4t(t - 2)(t + 2) \), so that \( t = 0 \), \( t = -2 \) and \( t = 2 \) are critical points. Next, \( s''(t) = 12t^2 - 16 \), so \( s''(-2) = 32 > 0 \), \( s''(0) = -16 < 0 \) and \( s''(2) = 32 > 0 \). Therefore, by the Second Derivative Test, \( s(0) \) is a local maximum while \( s(-2) \) and \( s(2) \) are local minima.

27. \( f(x) = x^2(x + 2)^3 \)

**SOLUTION**  Let \( f(x) = x^2(x + 2)^3 \). Then

\[
f'(x) = 3x^2(x + 2)^2 + 2x(x + 2)^3 = x(x + 2)^2(3x + 2x + 4) = x(x + 2)^2(5x + 4),
\]

so that \( x = 0 \), \( x = -2 \) and \( x = -\frac{4}{3} \) are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, \( f(-2) \) is neither a local maximum nor a local minimum, \( f(-\frac{4}{3}) \) is a local maximum and \( f(0) \) is a local minimum.

<table>
<thead>
<tr>
<th>Interval</th>
<th>((\infty, -2))</th>
<th>((-2, -\frac{4}{3}))</th>
<th>((-\frac{4}{3}, 0))</th>
<th>((0, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f' )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

28. \( f(x) = x^{2/3}(1 - x) \)

**SOLUTION**  Let \( f(x) = x^{2/3}(1 - x) = x^{2/3} - x^{5/3} \). Then

\[
f'(x) = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{2 - 5x}{3x^{1/3}},
\]

so that \( x = 0 \) and \( x = \frac{2}{3} \) are critical points. The sign of the first derivative on the intervals surrounding the critical points is indicated in the table below. Based on this information, \( f(0) \) is a local minimum and \( f(\frac{2}{3}) \) is a local maximum.

<table>
<thead>
<tr>
<th>Interval</th>
<th>((\infty, 0))</th>
<th>((0, \frac{2}{3}))</th>
<th>((\frac{2}{3}, \infty))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sign of ( f' )</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

29. \( g(\theta) = \sin^2 \theta + \theta \)

**SOLUTION**  Let \( g(\theta) = \sin^2 \theta + \theta \). Then

\[
g'(\theta) = 2 \sin \theta \cos \theta + 1 = 2 \sin 2\theta + 1,
\]

so the critical points are

\[
\theta = \frac{3\pi}{4} + n\pi
\]
for all integers \( n \). Because \( g'(\theta) \geq 0 \) for all \( \theta \), it follows that \( g \left( \frac{3\pi}{4} + n\pi \right) \) is neither a local maximum nor a local minimum for all integers \( n \).

30. \( h(\theta) = 2\cos\theta + \cos 4\theta \)

**SOLUTION** Let \( h(\theta) = 2\cos\theta + \cos 4\theta \). Then

\[
h'(\theta) = -4\sin 2\theta - 4 \sin 4\theta = -4\sin(2\theta + 2\cos 2\theta),
\]

so the critical points are

\[
\theta = \frac{n\pi}{2}, \quad \theta = \frac{\pi}{3} + \pi n \quad \text{and} \quad \theta = \frac{2\pi}{3} + \pi n
\]

for all integers \( n \). Now,

\[
h''(\theta) = -8\cos 2\theta - 16 \cos 4\theta,
\]

so

\[
h'' \left( \frac{n\pi}{2} \right) = -8\cos n\pi - 16 \cos 2n\pi = -8(-1)^n - 16 < 0;
\]

\[
h'' \left( \frac{\pi}{3} + n\pi \right) = -8\cos \frac{2\pi}{3} - 16 \cos \frac{4\pi}{3} = 12 > 0; \quad \text{and} \quad
\]

\[
h'' \left( \frac{2\pi}{3} + n\pi \right) = -8\cos \frac{4\pi}{3} - 16 \cos \frac{8\pi}{3} = 12 > 0,
\]

for all integers \( n \). Therefore, by the Second Derivative Test, \( h \left( \frac{n\pi}{2} \right) \) is a local maximum, and \( h \left( \frac{\pi}{3} + n\pi \right) \) and \( h \left( \frac{2\pi}{3} + n\pi \right) \) are local minima for all integers \( n \).

**In Exercises 31–38, find the extreme values on the interval.**

31. \( f(x) = x(10 - x), \quad [-1, 3] \)

**SOLUTION** Let \( f(x) = x(10 - x) = 10x - x^2 \). Then \( f'(x) = 10 - 2x \), so that \( x = 5 \) is the only critical point. As this critical point is not in the interval \([-1, 3]\), we only need to check the value of \( f \) at the endpoints to determine the extreme values. Because \( f(-1) = -11 \) and \( f(3) = 21 \), the maximum value of \( f(x) = x(10 - x) \) on the interval \([-1, 3]\) is 21 while the minimum value is \(-11\).

32. \( f(x) = 6x^4 - 4x^6, \quad [-2, 2] \)

**SOLUTION** Let \( f(x) = 6x^4 - 4x^6 \). Then \( f'(x) = 24x^3 - 24x^5 = 24x^3(1 - x^2) \), so that the critical points are \( x = -1, x = 0 \) and \( x = 1 \). The table below lists the value of \( f \) at each of the critical points and the endpoints of the interval \([-2, 2]\). Based on this information, the minimum value of \( f(x) = 6x^4 - 4x^6 \) on the interval \([-2, 2]\) is \(-170\) and the maximum value is 2.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>-170</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>-170</td>
</tr>
</tbody>
</table>

33. \( g(\theta) = \sin^2 \theta - \cos \theta, \quad [0, 2\pi] \)

**SOLUTION** Let \( g(\theta) = \sin^2 \theta - \cos \theta \). Then

\[
g'(\theta) = 2\sin \theta \cos \theta + \sin \theta = \sin(2\cos \theta + 1) = 0
\]

when \( \theta = 0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, 2\pi \). The table below lists the value of \( g \) at each of the critical points and the endpoints of the interval \([0, 2\pi]\). Based on this information, the minimum value of \( g(\theta) \) on the interval \([0, 2\pi]\) is \(-1\) and the maximum value is \( \frac{5}{4} \).

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>0</th>
<th>2\pi/3</th>
<th>\pi</th>
<th>4\pi/3</th>
<th>2\pi</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(\theta) )</td>
<td>-1</td>
<td>5/4</td>
<td>1</td>
<td>5/4</td>
<td>-1</td>
</tr>
</tbody>
</table>

34. \( R(t) = \sqrt{\frac{t}{t^2 + t + 1}}, \quad [0, 3] \)

**SOLUTION** Let \( R(t) = \frac{t}{t^2 + t + 1} \). Then

\[
R'(t) = \frac{t^2 + t + 1 - t(2t + 1)}{(t^2 + t + 1)^2} = \frac{1 - t^2}{(t^2 + t + 1)^2},
\]

so that the critical points are \( t = \pm 1 \). Note that only \( t = 1 \) is on the interval \([0, 3]\). With \( R(0) = 0, R(1) = \frac{1}{3} \) and \( R(3) = \frac{3}{11} \), it follows that the minimum value of \( R(t) \) on the interval \([0, 3]\) is 0 and the maximum value is \( \frac{1}{3} \).
35. \( f(x) = x^{2/3} - 2x^{1/3}, \quad [-1, 3] \)

**SOLUTION** Let \( f(x) = x^{2/3} - 2x^{1/3} \). Then \( f'(x) = \frac{2}{3}x^{-1/3} - \frac{2}{3}x^{-2/3} = \frac{2}{3}x^{-2/3}(x^{1/3} - 1) \), so that the critical points are \( x = 0 \) and \( x = 1 \). With \( f(-1) = 3 \), \( f(0) = 0 \), \( f(1) = -1 \) and \( f(3) = \sqrt[3]{9} - 2\sqrt[3]{3} \approx -0.804 \), it follows that the minimum value of \( f(x) \) on the interval \([-1, 3]\) is \(-1\) and the maximum value is \(3\).

36. \( f(x) = 4x - \tan^2 x, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \)

**SOLUTION** Let \( f(x) = 4x - \tan^2 x \). Then \( f'(x) = 4 - 2\tan x \sec^2 x \), and \( f''(x) = 0 \) when \( \tan x \sec^2 x = 2 \), so \( x = \frac{\pi}{4} \) is a solution. Since both sec \( x \) and \( x \) are positive and increasing on the given interval, it is the only solution, so that \( x = \frac{\pi}{4} \) is the only critical point on \( \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \). With \( f(-\frac{\pi}{4}) = 4(-\frac{\pi}{4}) - \tan^2(-\frac{\pi}{4}) = -\pi - 1 \), \( f(\frac{\pi}{4}) = 4(\frac{\pi}{4}) - \tan^2(\frac{\pi}{4}) = \frac{4\pi}{4} - 3 \), and \( f(\frac{\pi}{2}) = 4(\frac{\pi}{2}) - \tan^2(\frac{\pi}{2}) = \pi - 1 \), the minimum value is \(-\pi - 1 \approx -4.1416\) and the maximum value is \(\pi - 1 \approx 2.1416\).

37. \( f(x) = -x - 12 \ln x, \quad [5, 40] \)

**SOLUTION** Let \( f(x) = -x - 12 \ln x \). Then \( f'(x) = 1 - \frac{12}{x} \), whence \( x = 12 \) is the only critical point. The minimum value of \( f \) is then \( 12 - 12 \ln 12 \approx -17.818880 \), and the maximum value is \( 40 - 12 \ln 40 \approx -4.266553 \). Note that \( f(5) = 5 - 12 \ln 5 \approx -14.313255 \).

38. \( f(x) = e^x - 20x - 1, \quad [0, 5] \)

**SOLUTION** Let \( f(x) = e^x - 20x - 1 \). Then \( f'(x) = e^x - 20 \), whence \( x = \ln 20 \) is the only critical point. The minimum value of \( f \) is then \( 20 - 20 \ln 20 - 1 \approx -40.914645 \), and the maximum value is \( e^5 - 101 \approx 47.413159 \). Note that \( f(0) = 0 \).

39. Find the critical points and extreme values of \( f(x) = |x - 1| + |2x - 6| \) in \([0, 8]\).

**SOLUTION** Let

\[ f(x) = |x - 1| + |2x - 6| = \begin{cases} 7 - 3x, & x < 1 \\ 5 - x, & 1 \leq x < 3 \\ 3x - 7, & x \geq 3 \end{cases} \]

The derivative of \( f(x) \) is never zero but does not exist at the transition points \( x = 1 \) and \( x = 3 \). Thus, the critical points of \( f \) are \( x = 1 \) and \( x = 3 \). With \( f(0) = 7 \), \( f(1) = 4 \), \( f(3) = 2 \) and \( f(8) = 17 \), it follows that the minimum value of \( f(x) \) on the interval \([0, 8]\) is \(2\) and the maximum value is \(17\).

40. Match the description of \( f(x) \) with the graph of its derivative \( f'(x) \) in Figure 1.

(a) \( f(x) \) is increasing and concave up.

(b) \( f(x) \) is decreasing and concave up.

(c) \( f(x) \) is increasing and concave down.

![Graphs of the derivative.](image)

**FIGURE 1** Graphs of the derivative.

**SOLUTION**

(a) If \( f(x) \) is increasing and concave up, then \( f'(x) \) is positive and increasing. This matches the graph in (ii).

(b) If \( f(x) \) is decreasing and concave up, then \( f'(x) \) is negative and increasing. This matches the graph in (i).

(c) If \( f(x) \) is increasing and concave down, then \( f'(x) \) is positive and decreasing. This matches the graph in (iii).

In Exercises 41–46, find the points of inflection.

41. \( y = x^3 - 4x^2 + 4x \)

**SOLUTION** Let \( y = x^3 - 4x^2 + 4x \). Then \( y' = 3x^2 - 8x + 4 \) and \( y'' = 6x - 8 \). Thus, \( y'' > 0 \) and \( y \) is concave up for \( x > \frac{4}{3} \), while \( y'' < 0 \) and \( y \) is concave down for \( x < \frac{4}{3} \). Hence, there is a point of inflection at \( x = \frac{4}{3} \).

42. \( y = x - 2 \cos x \)

**SOLUTION** Let \( y = x - 2 \cos x \). Then \( y' = 1 + 2 \sin x \) and \( y'' = 2 \cos x \). Thus, \( y'' > 0 \) and \( y \) is concave up on each interval of the form \( \left(\frac{(4n-1)\pi}{2}, \frac{(4n+1)\pi}{2}\right) \),
while \( y'' < 0 \) and \( y \) is concave down on each interval of the form

\[
\left( \frac{(4n + 1)\pi}{2}, \frac{(4n + 3)\pi}{2} \right),
\]

where \( n \) is any integer. Hence, there is a point of inflection at

\[
x = \frac{(2n + 1)\pi}{2}
\]

for each integer \( n \).

43. \( y = \frac{x^2}{x^2 + 4} \)

**SOLUTION** Let \( y = \frac{x^2}{x^2 + 4} \) be \( 1 - \frac{4}{x^2 + 4} \). Then \( y' = \frac{8x}{(x^2 + 4)^2} \) and

\[
y'' = \frac{(x^2 + 4)^2(8) - 8x(2)(2x)(x^2 + 4)}{(x^2 + 4)^4} = \frac{8(4 - 3x^2)}{(x^2 + 4)^3}.
\]

Thus, \( y'' > 0 \) and \( y \) is concave up for

\[
-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}},
\]

while \( y'' < 0 \) and \( y \) is concave down for

\[
|x| > \frac{2}{\sqrt{3}}.
\]

Hence, there are points of inflection at

\[
x = \pm \frac{2}{\sqrt{3}}.
\]

44. \( y = \frac{x}{(x^2 - 4)^{1/3}} \)

**SOLUTION** Let \( y = \frac{x}{(x^2 - 4)^{1/3}} \). Then

\[
y' = \frac{(x^2 - 4)^{1/3} - \frac{1}{3}x(x^2 - 4)^{-2/3}(2x)}{(x^2 - 4)^{2/3}} = \frac{1}{3} \frac{x^2 - 12}{(x^2 - 4)^{4/3}}
\]

and

\[
y'' = \frac{1}{3} \frac{(x^2 - 4)^{4/3}(2x) - (x^2 - 12)^2(2x)(x^2 - 4)^{1/3}(2x)}{(x^2 - 4)^{8/3}} = \frac{2x(36 - x^2)}{9(x^2 - 4)^{7/3}}.
\]

Thus, \( y'' > 0 \) and \( y \) is concave up for \( x < -6, -2 < x < 0, 2 < x < 6 \), while \( y'' < 0 \) and \( y \) is concave down for \( -6 < x < -2, 0 < x < 2, x > 6 \). Hence, there are points of inflection at \( x = \pm 6 \) and \( x = 0 \). Note that \( x = \pm 2 \) are not points of inflection because these points are not in the domain of the function.

45. \( f(x) = (x^2 - x)e^{-x} \)

**SOLUTION** Let \( f(x) = (x^2 - x)e^{-x} \). Then

\[
y' = -(x^2 - x)e^{-x} + (2x - 1)e^{-x} = -(x^2 - 3x + 1)e^{-x}.
\]

and

\[
y'' = (x^2 - 3x + 1)e^{-x} + (2x - 3)e^{-x} = e^{-x}(x^2 - 5x + 4) = e^{-x}(x - 1)(x - 4).
\]

Thus, \( y'' > 0 \) and \( y \) is concave up for \( x < 1 \) and for \( x > 4 \), while \( y'' < 0 \) and \( y \) is concave down for \( 1 < x < 4 \). Hence, there are points of inflection at \( x = 1 \) and \( x = 4 \).

46. \( f(x) = x(\ln x)^2 \)

**SOLUTION** Let \( f(x) = x(\ln x)^2 \). Then

\[
y' = x \cdot 2 \ln x \cdot \frac{1}{x} + (\ln x)^2 = 2 \ln x + (\ln x)^2.
\]
and

\[ y'' = \frac{2}{x} + \frac{2}{x} \ln x = \frac{2}{x}(1 + \ln x). \]

Thus, \( y'' > 0 \) and \( y \) is concave up for \( x > \frac{1}{e} \), while \( y'' < 0 \) and \( y \) is concave down for \( 0 < x < \frac{1}{e} \). Hence, there is a point of inflection at \( x = \frac{1}{e} \).

In Exercises 47–56, sketch the graph, noting the transition points and asymptotic behavior.

47. \( y = 12x - 3x^2 \)

**SOLUTION** Let \( y = 12x - 3x^2 \). Then \( y' = 12 - 6x \) and \( y'' = -6 \). It follows that the graph of \( y = 12x - 3x^2 \) is increasing for \( x < 2 \), decreasing for \( x > 2 \), has a local maximum at \( x = 2 \) and is concave down for all \( x \). Because

\[ \lim_{x \to \pm \infty} (12x - 3x^2) = -\infty, \]

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

![Graph of y = 12x - 3x^2](image1)

48. \( y = 8x^2 - x^4 \)

**SOLUTION** Let \( y = 8x^2 - x^4 \). Then \( y' = 16x - 4x^3 = 4x(4 - x^2) \) and \( y'' = 16 - 12x^2 = 4(4 - 3x^2) \). It follows that the graph of \( y = 8x^2 - x^4 \) is increasing for \( x < -2 \) and \( 0 < x < 2 \), decreasing for \( -2 < x < 0 \) and \( x > 2 \), has a local maxima at \( x = \pm 2 \), has a local minimum at \( x = 0 \), is concave down for \( |x| > 2/\sqrt{3} \), is concave up for \( |x| < 2/\sqrt{3} \) and has inflection points at \( x = \pm 2/\sqrt{3} \). Because

\[ \lim_{x \to \pm \infty} (8x^2 - x^4) = -\infty, \]

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

![Graph of y = 8x^2 - x^4](image2)

49. \( y = x^3 - 2x^2 + 3 \)

**SOLUTION** Let \( y = x^3 - 2x^2 + 3 \). Then \( y' = 3x^2 - 4x \) and \( y'' = 6x - 4 \). It follows that the graph of \( y = x^3 - 2x^2 + 3 \) is increasing for \( x < 0 \) and \( x > \frac{2}{3} \), is decreasing for \( 0 < x < \frac{2}{3} \), has a local maximum at \( x = 0 \), has a local minimum at \( x = \frac{4}{3} \), is concave up for \( x > \frac{2}{3} \), is concave down for \( x < \frac{2}{3} \) and has a point of inflection at \( x = \frac{2}{3} \). Because

\[ \lim_{x \to -\infty} (x^3 - 2x^2 + 3) = -\infty \quad \text{and} \quad \lim_{x \to \infty} (x^3 - 2x^2 + 3) = \infty, \]

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

![Graph of y = x^3 - 2x^2 + 3](image3)

50. \( y = 4x - x^{3/2} \)
SOLUTION Let \( y = 4x - x^{3/2} \). First note that the domain of this function is \( x \geq 0 \). Now, \( y' = 4 - \frac{3}{2}x^{1/2} \) and \( y'' = -\frac{3}{4}x^{-1/2} \).

It follows that the graph of \( y = 4x - x^{3/2} \) is increasing for \( 0 < x < \frac{64}{3} \), is decreasing for \( x > \frac{64}{3} \), has a local maximum at \( x = \frac{64}{3} \) and is concave down for all \( x > 0 \).

\[
\lim_{x \to \infty} (4x - x^{3/2}) = -\infty,
\]

the graph has no horizontal asymptotes. There are also no vertical asymptotes. The graph is shown below.

51. \( y = \frac{x}{x^3 + 1} \)

SOLUTION Let \( y = \frac{x}{x^3 + 1} \). Then

\[
y' = \frac{x^3 + 1 - x(3x^2)}{(x^3 + 1)^2} = \frac{1 - 2x^3}{(x^3 + 1)^2}
\]

and

\[
y'' = \frac{(x^3 + 1)^2(-6x^2) - (1 - 2x^3)(2)(x^3 + 1)(3x^2)}{(x^3 + 1)^4} = -\frac{6x^2(2 - x^3)}{(x^3 + 1)^3}.
\]

It follows that the graph of \( y = \frac{x}{x^3 + 1} \) is increasing for \( x < -1 \) and \(-1 < x < \frac{3}{\sqrt{2}} \), is decreasing for \( x > \frac{3}{\sqrt{2}} \), has a local maximum at \( x = \frac{3}{\sqrt{2}} \) is concave up for \( x < -1 \) and \( x > \frac{3}{\sqrt{2}} \), is concave down for \(-1 < x < 0 \) and \( 0 < x < \frac{3}{\sqrt{2}} \) and has a point of inflection at \( x = \frac{3}{\sqrt{2}} \). Note that \( x = -1 \) is not an inflection point because \( x = -1 \) is not in the domain of the function. Now,

\[
\lim_{x \to \pm \infty} \frac{x}{x^3 + 1} = 0,
\]

so \( y = 0 \) is a horizontal asymptote. Moreover,

\[
\lim_{x \to -1^-} \frac{x}{x^3 + 1} = \infty \quad \text{and} \quad \lim_{x \to -1^+} \frac{x}{x^3 + 1} = -\infty,
\]

so \( x = -1 \) is a vertical asymptote. The graph is shown below.

52. \( y = \frac{x}{(x^2 - 4)^{2/3}} \)

SOLUTION Let \( y = \frac{x}{(x^2 - 4)^{2/3}} \). Then

\[
y' = \frac{(x^2 - 4)^{2/3} - \frac{2}{3}x(x^2 - 4)^{-1/3}(2x)}{(x^2 - 4)^{4/3}} = -\frac{1}{3} \frac{x^2 + 12}{(x^2 - 4)^{5/3}}
\]

and

\[
y'' = -\frac{1}{3} \frac{(x^2 - 4)^{5/3}(2x) - (x^2 + 12)(x^2 - 4)^{2/3}(2x)}{(x^2 - 4)^{10/3}} = \frac{4x(x^2 + 36)}{9(x^2 - 4)^{8/3}}.
\]

It follows that the graph of \( y = \frac{x}{(x^2 - 4)^{2/3}} \) is increasing for \(-2 < x < 2 \), is decreasing for \( |x| > 2 \), has no local extreme values, is concave up for \( 0 < x < 2, x > 2 \), is concave down for \( x < -2, -2 < x < 0 \) and has a point of inflection at \( x = 0 \). Note that \( x = \pm 2 \) are neither local extreme values nor inflection points because \( x = \pm 2 \) are not in the domain of the function. Now,

\[
\lim_{x \to \pm \infty} \frac{x}{(x^2 - 4)^{2/3}} = 0,
\]
so $y = 0$ is a horizontal asymptote. Moreover,
\[
\lim_{x \to -2^-} \frac{x}{(x^2 - 4)^{2/3}} = -\infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x}{(x^2 - 4)^{2/3}} = -\infty
\]
while
\[
\lim_{x \to -2^-} \frac{x}{(x^2 - 4)^{2/3}} = \infty \quad \text{and} \quad \lim_{x \to -2^+} \frac{x}{(x^2 - 4)^{2/3}} = \infty,
\]
so $x = \pm 2$ are vertical asymptotes. The graph is shown below.

53. $y = \frac{1}{|x + 2| + 1}$

**SOLUTION** Let $y = \frac{1}{|x + 2| + 1}$. Because
\[
\lim_{x \to \pm \infty} \frac{1}{|x + 2| + 1} = 0,
\]
the graph of this function has a horizontal asymptote of $y = 0$. The graph has no vertical asymptotes as $|x + 2| + 1 \geq 1$ for all $x$. The graph is shown below. From this graph we see there is a local maximum at $x = -2$.

54. $y = \sqrt{2 - x^3}$

**SOLUTION** Let $y = \sqrt{2 - x^3}$. Note that the domain of this function is $x \leq \sqrt[3]{2}$. Moreover, the graph has no vertical and no horizontal asymptotes. With
\[
y' = \frac{1}{2}(2 - x^3)^{-1/2}(-3x^2) = -\frac{3x^2}{2\sqrt{2 - x^3}}
\]
and
\[
y'' = \frac{1}{2}(2 - x^3)^{-1/2}(-6x) - \frac{3}{4}x^2(2 - x^3)^{-3/2}(3x^2) = \frac{3x(x^3 - 8)}{4(2 - x^3)^{3/2}},
\]
it follows that the graph of $y = \sqrt{2 - x^3}$ is decreasing over its entire domain, is concave up for $x < 0$, is concave down for $0 < x < \sqrt[3]{2}$ and has a point of inflection at $x = 0$. The graph is shown below.

55. $y = \sqrt[3]{\sin x - \cos x}$ on $[0, 2\pi]$

**SOLUTION** Let $y = \sqrt[3]{\sin x - \cos x}$. Then $y' = \sqrt[3]{\cos x + \sin x}$ and $y'' = -\sqrt[3]{\sin x + \cos x}$. It follows that the graph of $y = \sqrt[3]{\sin x - \cos x}$ is increasing for $0 < x < 5\pi/6$ and $11\pi/6 < x < 2\pi$, is decreasing for $5\pi/6 < x < 11\pi/6$, has a local maximum at $x = 5\pi/6$, has a local minimum at $x = 11\pi/6$, is concave up for $0 < x < \pi/3$ and $4\pi/3 < x < 2\pi$, is concave down for $\pi/3 < x < 4\pi/3$ and has points of inflection at $x = \pi/3$ and $x = 4\pi/3$. The graph is shown below.
56. \( y = 2x - \tan x \) on \([0, 2\pi]\)

**SOLUTION** Let \( y = 2x - \tan x \). Then \( y' = 2 - \sec^2 x \) and \( y'' = -2 \sec^2 x \tan x \). It follows that the graph of \( y = 2x - \tan x \) is increasing for \( 0 < x < \pi/4, 3\pi/4 < x < 5\pi/4, 7\pi/4 < x < 2\pi \), is decreasing for \( \pi/4 < x < \pi/2, \pi/2 < x < 3\pi/4, 5\pi/4 < x < 3\pi/2, 3\pi/2 < x < 7\pi/4 \), has local minima at \( x = 3\pi/4 \) and \( x = 7\pi/4 \), has local maxima at \( x = \pi/4 \) and \( x = 5\pi/4 \), is concave up for \( \pi/2 < x < \pi \) and \( 3\pi/2 < x < 2\pi \), is concave down for \( 0 < x < \pi/2 \) and \( \pi < x < 3\pi/2 \) and has an inflection point at \( x = \pi \). Moreover, because

\[
\lim_{x \to \pi/2^-} (2x - \tan x) = -\infty \quad \text{and} \quad \lim_{x \to \pi/2^+} (2x - \tan x) = \infty,
\]

while

\[
\lim_{x \to 3\pi/2^-} (2x - \tan x) = -\infty \quad \text{and} \quad \lim_{x \to 3\pi/2^+} (2x - \tan x) = \infty,
\]

the graph has vertical asymptotes at \( x = \pi/2 \) and \( x = 3\pi/2 \). The graph is shown below.

57. Draw a curve \( y = f(x) \) for which \( f' \) and \( f'' \) have signs as indicated in Figure 2.

**SOLUTION** The figure below depicts a curve for which \( f'(x) \) and \( f''(x) \) have the required signs.

58. Find the dimensions of a cylindrical can with a bottom but no top of volume \( 4 \text{ m}^3 \) that uses the least amount of metal.

**SOLUTION** Let the cylindrical can have height \( h \) and radius \( r \). Then

\[
V = \pi r^2 h = 4 \quad \text{so} \quad h = \frac{4}{\pi r^2}.
\]

The amount of metal needed to make the can is then

\[
M = 2\pi rh + \pi r^2 = \frac{8}{r} + \pi r^2.
\]

Now,

\[
M'(r) = -\frac{8}{r^2} + 2\pi r = 0 \quad \text{when} \quad r = \sqrt[3]{\frac{4}{\pi}}.
\]

Because \( M \to \infty \) as \( r \to 0^+ \) and as \( r \to \infty \), \( M \) must achieve its minimum for

\[
r = \sqrt[3]{\frac{4}{\pi}} \text{ m}.
\]
The height of the can is
\[ h = \frac{4}{\pi r^2} = \frac{4}{\sqrt[3]{\pi}} \text{ m}. \]

59. A rectangular box of height \( h \) with square base of side \( b \) has volume \( V = 4 \text{ m}^3 \). Two of the side faces are made of material costing $40/\text{m}^2$. The remaining sides cost $20/\text{m}^2$. Which values of \( b \) and \( h \) minimize the cost of the box?

**SOLUTION** Because the volume of the box is
\[ V = b^2 h = 4 \] it follows that \( h = \frac{4}{b^2} \).

Now, the cost of the box is
\[ C = 40(2bh) + 20(2bh) + 20b^2 = 120bh + 20b^2 = \frac{480}{b} + 20b^2. \]

Thus,
\[ C'(b) = -\frac{480}{b^2} + 40b = 0 \]
when \( b = \sqrt[3]{12} \) meters. Because \( C(b) \to \infty \) as \( b \to 0^+ \) and as \( b \to \infty \), it follows that cost is minimized when \( b = \sqrt[3]{12} \) meters and \( h = \frac{2}{\sqrt[3]{12}} \) meters.

60. The corn yield on a certain farm is
\[ Y = -0.118x^2 + 8.5x + 12.9 \] (bushels per acre)
where \( x \) is the number of corn plants per acre (in thousands). Assume that corn seed costs $1.25 (per thousand seeds) and that corn can be sold for $1.50/bushel. Let \( P(x) \) be the profit (revenue minus the cost of seeds) at planting level \( x \).

(a) Compute \( P(x_0) \) for the value \( x_0 \) that maximizes yield \( Y \).

(b) Find the maximum value of \( P(x) \). Does maximum yield lead to maximum profit?

**SOLUTION**

(a) Let \( Y = -0.118x^2 + 8.5x + 12.9 \). Then \( Y' = -0.236x + 8.5 = 0 \) when
\[ x_0 = \frac{8.5}{0.236} = 36.017 \text{ thousand corn plants/acre}. \]

Because \( Y'' = -0.236 < 0 \) for all \( x, x_0 \) corresponds to a maximum value for \( Y \). Thus, yield is maximized for a planting level of 36,017 corn plants per acre. At this planting level, the profit is
\[ 1.5Y(x_0) - 1.25x_0 = 1.5(165.972) - 1.25(36.017) = 203.94 \text{/acre}. \]

(b) As a function of planting level \( x \), the profit is
\[ P(x) = 1.5Y(x) - 1.25x = -0.177x^2 + 11.5x + 19.35. \]

Then, \( P'(x) = -0.354x + 11.5 = 0 \) when
\[ x_1 = \frac{11.5}{0.354} = 32.486 \text{ thousand corn plants/acre}. \]

Because \( P''(x) = -0.354 < 0 \) for all \( x, x_1 \) corresponds to a maximum value for \( P \). Thus, profit is maximized for a planting level of 32,486 corn plants per acre. Note the planting levels obtained in parts (a) and (b) are different. Thus, a maximum yield does not lead to maximum profit.

61. Let \( N(t) \) be the size of a tumor (in units of \( 10^6 \) cells) at time \( t \) (in days). According to the **Gompertz Model**, \( dN/dt = N(a - b \ln N) \) where \( a, b \) are positive constants. Show that the maximum value of \( N \) is \( e^a/b \) and that the tumor increases most rapidly when \( N = e^a/b - 1 \).

**SOLUTION** Given \( dN/dt = N(a - b \ln N) \), the critical points of \( N \) occur when \( N = 0 \) and when \( N = e^{a/b} \). The sign of \( N'(t) \) changes from positive to negative at \( N = e^{a/b} \) so the maximum value of \( N \) is \( e^{a/b} \). To determine when \( N \) changes most rapidly, we calculate
\[ N''(t) = N \left( \frac{b}{N} \right) + a - b \ln N = (a - b) - b \ln N. \]

Thus, \( N'(t) \) is increasing for \( N < e^{a/b - 1} \), is decreasing for \( N > e^{a/b - 1} \) and is therefore maximum when \( N = e^{a/b - 1} \). Therefore, the tumor increases most rapidly when \( N = e^{a/b - 1} \).
62. A truck gets 10 miles per gallon of diesel fuel traveling along an interstate highway at 50 mph. This mileage decreases by 0.15 mpg for each mile per hour increase above 50 mph.

(a) If the truck driver is paid $30/hour and diesel fuel costs \( P = \$3 \)/gal, which speed \( v \) between 50 and 70 mph will minimize the cost of a trip along the highway? Notice that the actual cost depends on the length of the trip, but the optimal speed does not.

(b) Plot cost as a function of \( v \) (choose the length arbitrarily) and verify your answer to part (a).

(c) Do you expect the optimal speed \( v \) to increase or decrease if fuel costs go down to \( P = \$2 \)/gal? Plot the graphs of cost as a function of \( v \) for \( P = 2 \) and \( P = 3 \) on the same axis and verify your conclusion.

**SOLUTION**

(a) If the truck travels \( L \) miles at a speed of \( v \) mph, then the time required is \( L/v \), and the wages paid to the driver are \( 30L/v \). The cost of the fuel is

\[
\frac{3L}{10 - 0.15(v - 50)} = \frac{3L}{17.5 - 0.15v}.
\]

the total cost is therefore

\[
C(v) = \frac{30L}{v} + \frac{3L}{17.5 - 0.15v}.
\]

Solving

\[
C'(v) = L \left( \frac{30}{v^2} + \frac{0.45}{(17.5 - 0.15v)^2} \right) = 0
\]

yields

\[
v = \frac{175\sqrt{6}}{3 + 1.5\sqrt{6}} \approx 64.2 \text{ mph}.
\]

Because \( C(50) = 0.9L, C(64.2) \approx 0.848L \) and \( C(70) \approx 0.857L \), we see that the optimal speed is \( v \approx 64.2 \) mph.

(b) The cost as a function of speed is shown below for \( L = 100 \). The optimal speed is clearly around 64 mph.

(c) We expect \( v \) to increase if \( P \) goes down to \$2 per gallon. When gas is cheaper, it is better to drive faster and thereby save on the driver’s wages. The cost as a function of speed for \( P = 2 \) and \( P = 3 \) is shown below (with \( L = 100 \)). When \( P = 2 \), the optimal speed is \( v = 70 \) mph, which is an increase over the optimal speed when \( P = 3 \).

63. Find the maximum volume of a right-circular cone placed upside-down in a right-circular cone of radius \( R = 3 \) and height \( H = 4 \) as in Figure 3. A cone of radius \( r \) and height \( h \) has volume \( \frac{1}{3}\pi r^2h \).
SOLUTION Let \( r \) denote the radius and \( h \) the height of the upside down cone. By similar triangles, we obtain the relation

\[
\frac{4 - h}{r} = \frac{4}{3} \quad \text{so} \quad h = 4 \left(1 - \frac{r}{3}\right)
\]

and the volume of the upside down cone is

\[
V(r) = \frac{1}{3} \pi r^2 h = \frac{4}{3} \pi \left(r^2 - \frac{r^3}{3}\right)
\]

for \( 0 \leq r \leq 3 \). Thus,

\[
\frac{dV}{dr} = \frac{4}{3} \pi \left(2r - r^2\right),
\]

and the critical points are \( r = 0 \) and \( r = 2 \). Because \( V(0) = V(3) = 0 \) and

\[
V(2) = \frac{4}{3} \pi \left(4 - \frac{8}{3}\right) = \frac{16}{9} \pi.
\]

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius 3 and height 4 is

\[
\frac{16}{9} \pi.
\]

64. Redo Exercise 63 for arbitrary \( R \) and \( H \).

SOLUTION Let \( r \) denote the radius and \( h \) the height of the upside down cone. By similar triangles, we obtain the relation

\[
\frac{H - h}{r} = \frac{H}{R} \quad \text{so} \quad h = H \left(1 - \frac{r}{R}\right)
\]

and the volume of the upside down cone is

\[
V(r) = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi H \left(r^2 - \frac{r^3}{R}\right)
\]

for \( 0 \leq r \leq R \). Thus,

\[
\frac{dV}{dr} = \frac{1}{3} \pi H \left(2r - \frac{3r^2}{R}\right),
\]

and the critical points are \( r = 0 \) and \( r = 2R/3 \). Because \( V(0) = V(R) = 0 \) and

\[
V \left(\frac{2R}{3}\right) = \frac{1}{3} \pi H \left(\frac{4R^2}{9} - \frac{8R^2}{27}\right) = \frac{4}{81} \pi R^2 H.
\]

the maximum volume of a right-circular cone placed upside down in a right-circular cone of radius \( R \) and height \( H \) is

\[
\frac{4}{81} \pi R^2 H.
\]

65. Show that the maximum area of a parallelogram \( ADEF \) that is inscribed in a triangle \( ABC \), as in Figure 4, is equal to one-half the area of \( \triangle ABC \).

SOLUTION Let \( \theta \) denote the measure of angle \( BAC \). Then the area of the parallelogram is given by \( \overline{AD} \cdot \overline{AF} \sin \theta \). Now, suppose that

\[
\overline{BE}/\overline{BC} = x.
\]

Then, by similar triangles, \( \overline{AD} = (1 - x)\overline{AB}, \overline{AF} = \overline{DE} = x\overline{AC} \), and the area of the parallelogram becomes \( \overline{AB} \cdot \overline{AC} x(1 - x) \sin \theta \). The function \( x(1 - x) \) achieves its maximum value of \( \frac{1}{4} \) when \( x = \frac{1}{2} \). Thus, the maximum area of a parallelogram inscribed in a triangle \( \triangle ABC \) is

\[
\frac{1}{4} \overline{AB} \cdot \overline{AC} \sin \theta = \frac{1}{2} \left(\frac{1}{2} \overline{AB} \cdot \overline{AC} \sin \theta\right) = \frac{1}{2} \text{ (area of } \triangle ABC)\right).
\]
66. A box of volume 8 m³ with a square top and bottom is constructed out of two types of metal. The metal for the top and bottom costs $50/m² and the metal for the sides costs $30/m². Find the dimensions of the box that minimize total cost.

**SOLUTION** Let the square base have side length $s$ and the box have height $h$. Then

$$V = s^2 h = 8 \quad \text{so} \quad h = \frac{8}{s^2}.$$  

The cost of the box is then

$$C = 100s^2 + 120sh = 100s^2 + \frac{960}{s}.$$  

Now,

$$C'(s) = 200s - \frac{960}{s^2} = 0 \quad \text{when} \quad s = \frac{3}{\sqrt{4.8}}.$$  

Because $C(s) \to \infty$ as $s \to 0+$ and as $s \to \infty$, it follows that total cost is minimized when $s = \frac{3}{\sqrt{4.8}} \approx 1.69$ meters. The height of the box is

$$h = \frac{8}{s^2} \approx 2.81 \text{ meters.}$$

67. Let $f(x)$ be a function whose graph does not pass through the $x$-axis and let $Q = (a, 0)$. Let $P = (x_0, f(x_0))$ be the point on the graph closest to $Q$ (Figure 5). Prove that $PQ$ is perpendicular to the tangent line to the graph of $x_0$. **Hint:** Find the minimum value of the square of the distance from $(x, f(x))$ to $(a, 0)$.

**SOLUTION** Let $P = (a, 0)$ and let $Q = (x_0, f(x_0))$ be the point on the graph of $y = f(x)$ closest to $P$. The slope of the segment joining $P$ and $Q$ is then

$$\frac{f(x_0)}{x_0 - a}.$$  

Now, let

$$q(x) = \sqrt{(x - a)^2 + (f(x))^2},$$  

the distance from the arbitrary point $(x, f(x))$ on the graph of $y = f(x)$ to the point $P$. As $(x_0, f(x_0))$ is the point closest to $P$, we must have

$$q'(x_0) = \frac{2(x_0 - a) + 2f(x_0)f'(x_0)}{\sqrt{(x_0 - a)^2 + (f(x_0))^2}} = 0.$$  

Thus,

$$f'(x_0) = -\frac{x_0 - a}{f(x)} = -\left(\frac{f(x_0)}{x_0 - a}\right)^{-1}.$$  

In other words, the slope of the segment joining $P$ and $Q$ is the negative reciprocal of the slope of the line tangent to the graph of $y = f(x)$ at $x = x_0$; hence; the two lines are perpendicular.

68. Take a circular piece of paper of radius $R$, remove a sector of angle $\theta$ (Figure 6), and fold the remaining piece into a cone-shaped cup. Which angle $\theta$ produces the cup of largest volume?
SOLUTION Let \( r \) denote the radius and \( h \) denote the height of the cone-shaped cup. Having removed an angle of \( \theta \) from the paper, there is an arc of length \((2\pi - \theta)R\) remaining to form the circumference of the cup; hence

\[
r = \frac{(2\pi - \theta)R}{2\pi} = \left(1 - \frac{\theta}{2\pi}\right)R.
\]

The height of the cup is then

\[
h = \sqrt{R^2 - \left(1 - \frac{\theta}{2\pi}\right)^2} = R\sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2},
\]

and the volume of the cup is

\[
V(\theta) = \frac{1}{3} \pi R^3 \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}
\]

for \( 0 \leq \theta \leq 2\pi \). Now,

\[
\frac{dV}{d\theta} = 2 \left(1 - \frac{\theta}{2\pi}\right) (-1) \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2} + \left(1 - \frac{\theta}{2\pi}\right)^2 \left(2\pi \frac{\theta}{2\pi} \left(1 - \frac{\theta}{2\pi}\right)ight) \sqrt{1 - \left(1 - \frac{\theta}{2\pi}\right)^2}
\]

so that \( \theta = 2\pi \) and \( \theta = 2\pi \pm \frac{2\pi \sqrt{3}}{3} \) are critical points. With \( V(0) = V(2\pi) = 0 \) and

\[
V \left(2\pi - \frac{2\pi \sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{27} \pi R^3.
\]

the volume of the cup is maximized when \( \theta = 2\pi - \frac{2\pi \sqrt{3}}{3} \).

69. Use Newton’s Method to estimate \( \sqrt[3]{25} \) to four decimal places.

SOLUTION Let \( f(x) = x^3 - 25 \) and define

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 25}{3x_n^2}.
\]

With \( x_0 = 3 \), we find

<table>
<thead>
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<th>( n )</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.925925926</td>
</tr>
<tr>
<td>2</td>
<td>2.924018982</td>
</tr>
<tr>
<td>3</td>
<td>2.924017738</td>
</tr>
</tbody>
</table>

Thus, to four decimal places \( \sqrt[3]{25} = 2.9240 \).

70. Use Newton’s Method to find a root of \( f(x) = x^2 - x - 1 \) to four decimal places.

SOLUTION Let \( f(x) = x^2 - x - 1 \) and define

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1}.
\]

The graph below suggests the two roots of \( f(x) \) are located near \( x = -1 \) and \( x = 2 \).
With $x_0 = -1$, we find

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>-0.666666667</td>
<td>-0.619047619</td>
<td>-0.6180344477</td>
<td>-0.6180339889</td>
</tr>
</tbody>
</table>

On the other hand, with $x_0 = 2$, we find

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_n$</td>
<td>1.666666667</td>
<td>1.619047619</td>
<td>1.618034448</td>
<td>1.618033989</td>
</tr>
</tbody>
</table>

Thus, to four decimal places, the roots of $f(x) = x^2 - x - 1$ are $-0.6180$ and $1.6180$.

In Exercises 71–84, calculate the indefinite integral.

71. $\int (4x^3 - 2x^2) \, dx$

**SOLUTION**  
$\int (4x^3 - 2x^2) \, dx = x^4 - \frac{2}{3}x^3 + C$.

72. $\int x^{9/4} \, dx$

**SOLUTION**  
$\int x^{9/4} \, dx = \frac{4}{13}x^{13/4} + C$.

73. $\int \sin(\theta - 8) \, d\theta$

**SOLUTION**  
$\int \sin(\theta - 8) \, d\theta = -\cos(\theta - 8) + C$.

74. $\int \cos(5 - 7\theta) \, d\theta$

**SOLUTION**  
$\int \cos(5 - 7\theta) \, d\theta = \frac{1}{7}\sin(5 - 7\theta) + C$.

75. $\int (4t^{-3} - 12t^{-4}) \, dt$

**SOLUTION**  
$\int (4t^{-3} - 12t^{-4}) \, dt = -2t^{-2} + 4t^{-3} + C$.

76. $\int (9t^{-2/3} + 4t^{7/3}) \, dt$

**SOLUTION**  
$\int (9t^{-2/3} + 4t^{7/3}) \, dt = 27t^{1/3} + \frac{6}{5}t^{10/3} + C$.

77. $\int \sec^2 x \, dx$

**SOLUTION**  
$\int \sec^2 x \, dx = \tan x + C$.

78. $\int \tan 3\theta \sec 3\theta \, d\theta$

**SOLUTION**  
$\int \tan 3\theta \sec 3\theta \, d\theta = \frac{1}{3} \sec 3\theta + C$.

79. $\int (y + 2)^4 \, dy$

**SOLUTION**  
$\int (y + 2)^4 \, dy = \frac{1}{5}(y + 2)^5 + C$.

80. $\int \frac{3x^3 - 9}{x^2} \, dx$

**SOLUTION**  
$\int \frac{3x^3 - 9}{x^2} \, dx = \int (3x - 9x^{-2}) \, dx = \frac{3}{2}x^2 + 9x^{-1} + C$. 
81. \[ \int (e^x - x) \, dx \]

**SOLUTION** \[ \int (e^x - x) \, dx = e^x - \frac{1}{2}x^2 + C. \]

82. \[ \int e^{-4x} \, dx \]

**SOLUTION** \[ \int e^{-4x} \, dx = -\frac{1}{4}e^{-4x} + C. \]

83. \[ \int 4x^{-1} \, dx \]

**SOLUTION** \[ \int 4x^{-1} \, dx = 4 \ln |x| + C. \]

84. \[ \int \sin(4x - 9) \, dx \]

**SOLUTION** \[ \int \sin(4x - 9) \, dx = -\frac{1}{4} \cos(4x - 9) + C. \]

In Exercises 85–90, solve the differential equation with the given initial condition.

85. \[ \frac{dy}{dx} = 4x^3, \quad y(1) = 4 \]

**SOLUTION** Let \( \frac{dy}{dx} = 4x^3 \). Then

\[ y(x) = \int 4x^3 \, dx = x^4 + C. \]

Using the initial condition \( y(1) = 4 \), we find \( y(1) = 1 + C = 4 \), so \( C = 3 \). Thus, \( y(x) = x^4 + 3 \).

86. \[ \frac{dy}{dt} = 3t^2 + \cos t, \quad y(0) = 12 \]

**SOLUTION** Let \( \frac{dy}{dt} = 3t^2 + \cos t \). Then

\[ y(t) = \int (3t^2 + \cos t) \, dt = t^3 + \sin t + C. \]

Using the initial condition \( y(0) = 12 \), we find \( y(0) = 0^3 + 0 + C = 12 \), so \( C = 12 \). Thus, \( y(t) = t^3 + \sin t + 12 \).

87. \[ \frac{dy}{dx} = x^{-1/2}, \quad y(1) = 1 \]

**SOLUTION** Let \( \frac{dy}{dx} = x^{-1/2} \). Then

\[ y(x) = \int x^{-1/2} \, dx = 2x^{1/2} + C. \]

Using the initial condition \( y(1) = 1 \), we find \( y(1) = 2^{1/2} + C = 1 \), so \( C = -1 \). Thus, \( y(x) = 2x^{1/2} - 1 \).

88. \[ \frac{dy}{dx} = \sec^2 x, \quad y\left(\frac{\pi}{4}\right) = 2 \]

**SOLUTION** Let \( \frac{dy}{dx} = \sec^2 x \). Then

\[ y(x) = \int \sec^2 x \, dx = \tan x + C. \]

Using the initial condition \( y\left(\frac{\pi}{4}\right) = 2 \), we find \( y\left(\frac{\pi}{4}\right) = \tan \frac{\pi}{4} + C = 2 \), so \( C = 1 \). Thus, \( y(x) = \tan x + 1 \).

89. \[ \frac{dy}{dx} = e^{-x}, \quad y(0) = 3 \]

**SOLUTION** Let \( \frac{dy}{dx} = e^{-x} \). Then

\[ y(x) = \int e^{-x} \, dx = -e^{-x} + C. \]

Using the initial condition \( y(0) = 3 \), we find \( y(0) = -e^{0} + C = 3 \), so \( C = 4 \). Thus, \( y(x) = 4 - e^{-x} \).
90. \( \frac{dy}{dx} = e^{4x}, \ y(1) = 1 \)

**SOLUTION** Let \( \frac{dy}{dx} = e^{4x} \). Then

\[
y(x) = \int e^{4x} \, dx = \frac{1}{4} e^{4x} + C.
\]

Using the initial condition \( y(1) = 1 \), we find \( y(1) = \frac{1}{4} e^4 + C = 1 \), so \( C = 1 - \frac{1}{4} e^4 \). Thus, \( y(x) = \frac{1}{4} e^{4x} + 1 - \frac{1}{4} e^4 \).

91. Find \( f(t) \) if \( f''(t) = 1 - 2t, \ f(0) = 2, \) and \( f'(0) = -1 \).

**SOLUTION** Suppose \( f''(t) = 1 - 2t \). Then

\[
f'(t) = \int f''(t) \, dt = \int (1 - 2t) \, dt = t - t^2 + C.
\]

Using the initial condition \( f'(0) = -1 \), we find \( f'(0) = 0 - 0^2 + C = -1 \), so \( C = -1 \). Thus, \( f'(t) = t - t^2 - 1 \). Now,

\[
f(t) = \int f'(t) \, dt = \int (t - t^2 - 1) \, dt = \frac{1}{2} t^2 - \frac{1}{3} t^3 - t + C.
\]

Using the initial condition \( f(0) = 2 \), we find \( f(0) = \frac{1}{2} 0^2 - \frac{1}{3} 0^3 - 0 + C = 2 \), so \( C = 2 \). Thus,

\[
f(t) = \frac{1}{2} t^2 - \frac{1}{3} t^3 - t + 2.
\]

92. At time \( t = 0 \), a driver begins decelerating at a constant rate of \(-10 \text{ m/s}^2\) and comes to a halt after traveling 500 m. Find the velocity at \( t = 0 \).

**SOLUTION** From the constant deceleration of \(-10 \text{ m/s}^2\), we determine

\[
v(t) = \int (-10) \, dt = -10t + v_0,
\]

where \( v_0 \) is the velocity of the automobile at \( t = 0 \). Note the automobile comes to a halt when \( v(t) = 0 \), which occurs at

\[
t = \frac{v_0}{10} \text{ s}.
\]

The distance traveled during the braking process is

\[
s(t) = \int v(t) \, dt = -5t^2 + v_0 t + C,
\]

for some arbitrary constant \( C \). We are given that the braking distance is 500 meters, so

\[
s \left( \frac{v_0}{10} \right) - s(0) = -5 \left( \frac{v_0}{10} \right)^2 + v_0 \left( \frac{v_0}{10} \right) + C - C = 500,
\]

leading to

\[
v_0 = 100 \text{ m/s}.
\]

93. Find the local extrema of \( f(x) = \frac{e^{2x} + 1}{e^{x} + 1} \).

**SOLUTION** To simplify the differentiation, we first rewrite \( f(x) = \frac{e^{2x} + 1}{e^{x} + 1} \) using the Laws of Exponents:

\[
f(x) = \frac{e^{2x}}{e^x + 1} + \frac{1}{e^{x} + 1} = e^{2x-x} + e^{-(x+1)} = e^{x-1} + e^{-x-1}.
\]

Now,

\[
f'(x) = e^{x-1} - e^{-x-1}.
\]

Setting the derivative equal to zero yields

\[
e^{x-1} - e^{-x-1} = 0 \quad \text{or} \quad e^{x-1} = e^{-x-1}.
\]

Thus,

\[
x - 1 = -x - 1 \quad \text{or} \quad x = 0.
\]

Next, we use the Second Derivative Test. With \( f''(x) = e^{x-1} + e^{-x-1} \), it follows that

\[
f''(0) = e^{-1} + e^{-1} = \frac{2}{e} > 0.
\]

Hence, \( x = 0 \) is a local minimum. Since \( f(0) = e^{0-1} + e^{0-1} = \frac{2}{e} \), we conclude that the point \( (0, \frac{2}{e}) \) is a local minimum.
94. Find the points of inflection of \( f(x) = \ln(x^2 + 1) \), and at each point, determine whether the concavity changes from up to down or from down to up.

**SOLUTION** With \( f(x) = \ln(x^2 + 1) \), we find

\[
f'(x) = \frac{2x}{x^2 + 1} \quad \text{and} \quad f''(x) = \frac{2(2x) - 2x \cdot 2x}{(x^2 + 1)^2} = \frac{2(1 - x^2)}{(x^2 + 1)^2}.
\]

Thus, \( f''(x) > 0 \) for \(-1 < x < 1\), whereas \( f''(x) < 0 \) for \( x < -1 \) and for \( x > 1 \). It follows that there are points of inflection at \( x = \pm 1 \), and that the concavity of \( f \) changes from down to up at \( x = -1 \) and from up to down at \( x = 1 \).

In Exercises 95–98, find the local extrema and points of inflection, and sketch the graph. Use L'Hôpital's Rule to determine the limits as \( x \to 0^+ \) or \( x \to \pm \infty \) if necessary.

95. \( y = x \ln x \) (\( x > 0 \))

**SOLUTION** Let \( y = x \ln x \). Then

\[
y' = \ln x + x \left( \frac{1}{x} \right) = 1 + \ln x,
\]

and \( y'' = \frac{1}{x} \). Solving \( y' = 0 \) yields the critical point \( x = e^{-1} \). Since \( y''(e^{-1}) = e > 0 \), the function has a local minimum at \( x = e^{-1} \). \( y'' \) is positive for \( x > 0 \), hence the function is concave up for \( x > 0 \) and there are no points of inflection. As \( x \to 0^+ \) and as \( x \to \infty \), we find

\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} (-x) = 0; \\
\lim_{x \to \infty} x \ln x = \infty.
\]

The graph is shown below.

96. \( y = e^{x-x^2} \)

**SOLUTION** Let \( y = e^{x-x^2} \). Then \( y' = (1 - 2x)e^{x-x^2} \) and

\[
y'' = (2 - 4x^2 - 4x^2 - 2e^{x-x^2} = (4x^2 - 4x - 1)e^{x-x^2}.
\]

Solving \( y' = 0 \) yields the critical point \( x = \frac{1}{2} \). Since \( y'' \left( \frac{1}{2} \right) = -2e^{1/4} < 0 \), the function has a local maximum at \( x = \frac{1}{2} \). Using the quadratic formula, we find that \( y'' = 0 \) when \( x = \frac{1}{2} \pm \frac{1}{2} \sqrt{2} \). \( y'' > 0 \) and the function is concave up for \( x < \frac{1}{2} - \frac{1}{2} \sqrt{2} \) and for \( x > \frac{1}{2} + \frac{1}{2} \sqrt{2} \), whereas \( y'' < 0 \) and the function is concave down for \( \frac{1}{2} - \frac{1}{2} \sqrt{2} < x < \frac{1}{2} + \frac{1}{2} \sqrt{2} \); hence, there are inflection points at \( x = \frac{1}{2} \pm \frac{1}{2} \sqrt{2} \). As \( x \to \pm \infty \), \( x - x^2 \to -\infty \) so

\[
\lim_{x \to \pm \infty} e^{x-x^2} = 0.
\]

The graph is shown below.
97. \( y = x (\ln x)^2 \) \((x > 0)\)

**SOLUTION**  Let \( y = x (\ln x)^2 \). Then

\[
y' = x \frac{2 \ln x}{x} + (\ln x)^2 = 2 \ln x + (\ln x)^2 = \ln x (2 + \ln x),
\]

and

\[
y'' = \frac{2}{x} + \frac{2 \ln x}{x} = \frac{2}{x} (1 + \ln x).
\]

Solving \( y' = 0 \) yields the critical point \( x = e^{-2} \) and \( x = 1 \). Since \( y''(e^{-2}) = -2e^2 < 0 \) and \( y''(1) = 2 > 0 \), the function has a local maximum at \( x = e^{-2} \) and a local minimum at \( x = 1 \). \( y'' < 0 \) and the function is concave down for \( x < e^{-1} \), whereas \( y'' > 0 \) and the function is concave up for \( x > e^{-1} \); hence, there is a point of inflection at \( x = e^{-1} \). As \( x \to 0^+ \) and as \( x \to \infty \), we find

\[
\lim_{x \to 0^+} x (\ln x)^2 = \lim_{x \to 0^+} (\ln x)^2 = \lim_{x \to 0^+} \frac{2 \ln x \cdot x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} \frac{2 \ln x}{-x^{-1}} = \lim_{x \to 0^+} \frac{2x^{-1}}{-x^{-2}} = \lim_{x \to 0^+} 2x = 0;
\]

\[
\lim_{x \to \infty} x (\ln x)^2 = \infty.
\]

The graph is shown below:

![Graph of y = x (ln x)^2](image)

98. \( y = \tan^{-1} \left( \frac{x^2}{4} \right) \)

**SOLUTION**  Let \( y = \tan^{-1} \left( \frac{x^2}{4} \right) \). Then

\[
y' = \frac{1}{1 + \left( \frac{x^2}{4} \right)^2} \frac{x}{2} = \frac{8x}{x^4 + 16},
\]

and

\[
y'' = \frac{8(x^4 + 16) - 8x \cdot 4x^3}{(x^4 + 16)^2} = \frac{128 - 24x^4}{(x^4 + 16)^2}.
\]

Solving \( y' = 0 \) yields \( x = 0 \) as the only critical point. Because \( y''(0) = \frac{1}{2} > 0 \), we conclude the function has a local minimum at \( x = 0 \). Moreover, \( y'' < 0 \) for \( x > 2 \cdot \frac{1}{2} \) and for \( x > 2 \cdot \frac{1}{2} \), whereas \( y'' > 0 \) for \(-2 \cdot \frac{1}{2} < x < 2 \cdot \frac{1}{2} \). Therefore, there are points of inflection at \( x = \pm \frac{1}{2} \). As \( x \to \pm \infty \), we find

\[
\lim_{x \to \pm \infty} \tan^{-1} \left( \frac{x^2}{4} \right) = \frac{\pi}{2}.
\]

The graph is shown below:

![Graph of y = tan^{-1}(x^2/4)](image)

99. Explain why L'Hôpital's Rule gives no information about \( \lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} \). Evaluate the limit by another method.
SOLUTION As \( x \to \infty \), both \( 2x - \sin x \) and \( 3x + \cos 2x \) tend toward infinity, so L’Hôpital’s Rule applies to \( \lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} \); however, the resulting limit, \( \lim_{x \to \infty} \frac{2 - \cos x}{3 - 2 \sin 2x} \), does not exist due to the oscillation of \( \sin x \) and \( \cos x \) and further applications of L’Hôpital’s rule will not change this situation.

To evaluate the limit, we note
\[
\lim_{x \to \infty} \frac{2x - \sin x}{3x + \cos 2x} = \lim_{x \to \infty} \frac{2 - \sin \frac{x}{3}}{3 + \cos \frac{2x}{3}} = \frac{2}{3}.
\]

100. Let \( f(x) \) be a differentiable function with inverse \( g(x) \) such that \( f(0) = 0 \) and \( f'(0) \neq 0 \). Prove that
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = f'(0)^2.
\]

**SOLUTION** Since \( g \) and \( f \) are inverse functions, we have \( g(f(x)) = x \) for all \( x \) in the domain of \( f \). In particular, for \( x = 0 \) we have
\[
g(0) = g(f(0)) = 0.
\]

Therefore, the limit is an indeterminate form of type \( \frac{0}{0} \), so we may apply L’Hôpital’s Rule. By the Theorem on the derivative of the inverse function, we have
\[
g'(x) = \frac{1}{f'(g(x))}.
\]

Therefore,
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} f'(x)g'(x) = f'(0)g'(0) = f'(0) \cdot f'(0) = f'(0)^2.
\]

In Exercises 101–112, verify that L’Hôpital’s Rule applies and evaluate the limit.

101. \( \lim_{x \to 3} \frac{4x - 12}{x^2 - 5x + 6} \)

**SOLUTION** The given expression is an indeterminate form of type \( \frac{0}{0} \), therefore L’Hôpital’s Rule applies. We find
\[
\lim_{x \to 3} \frac{4x - 12}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{4}{2x - 5} = 4.
\]

102. \( \lim_{x \to -2} \frac{x^3 + 2x^2 - x - 2}{x^4 + 2x^3 - 4x - 8} \)

**SOLUTION** The given expression is an indeterminate form of type \( \frac{0}{0} \), therefore L’Hôpital’s Rule applies. We find
\[
\lim_{x \to -2} \frac{x^3 + 2x^2 - x - 2}{x^4 + 2x^3 - 4x - 8} = \lim_{x \to -2} \frac{3x^2 + 4x - 1}{4x^3 + 6x^2 - 4} = \frac{3}{12} = \frac{1}{4}.
\]

103. \( \lim_{x \to 0^+} \frac{1}{2} \ln x \)

**SOLUTION** First rewrite
\[
\frac{1}{2} \ln x = \frac{\ln x}{x^{-1/2}}.
\]

The rewritten expression is an indeterminate form of type \( \frac{\infty}{\infty} \), therefore L’Hôpital’s Rule applies. We find
\[
\lim_{x \to 0^+} \frac{1}{2} \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \to 0^+} \frac{1/x}{1/2x^{-3/2}} = \lim_{x \to 0^+} \frac{1/2}{2} = 0.
\]

104. \( \lim_{t \to \infty} \frac{\ln(e^t + 1)}{t} \)

**SOLUTION** The given expression is an indeterminate form of type \( \frac{\infty}{\infty} \); hence, we may apply L’Hôpital’s Rule. We find
\[
\lim_{t \to \infty} \frac{\ln(e^t + 1)}{t} = \lim_{t \to \infty} \frac{e^t}{e^t + 1} = \lim_{t \to \infty} \frac{1}{1 + e^{-t}} = 1.
\]
105. \[ \lim_{{\theta \to 0}} \frac{2 \sin \theta - \sin 2\theta}{\sin \theta - \cos \theta} \]

**SOLUTION** The given expression is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{\theta \to 0}} \frac{2 \sin \theta - \sin 2\theta}{\sin \theta - \cos \theta} = \lim_{{\theta \to 0}} \frac{2 \cos \theta - 2 \cos 2\theta}{\cos \theta - \sin \theta} = \lim_{{\theta \to 0}} \frac{2 \cos \theta - 2 \cos 2\theta}{\sin \theta} = \lim_{{\theta \to 0}} (2 \sin \theta + 4 \sin 2\theta) \cdot \frac{1}{\cos \theta} - \sin \theta ) = \frac{-2 \cos \theta + 8 \cos 2\theta}{1 + 1 - 0} = 3.
\]

106. \[ \lim_{{x \to 0}} \frac{\sqrt{4 + x} - 2 \sqrt{1 + x}}{x^2} \]

**SOLUTION** The given expression is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{x \to 0}} \frac{\sqrt{4 + x} - 2 \sqrt{1 + x}}{x^2} = \lim_{{x \to 0}} \frac{\frac{1}{2}(4 + x)^{-1/2}}{2x} - \frac{1}{2}(1 + x)^{-7/8} = \lim_{{x \to 0}} \frac{-\frac{1}{4}(4 + x)^{-3/2} + \frac{7}{32}(1 + x)^{-15/8}}{2} = \frac{-\frac{1}{4} \cdot \frac{1}{8} + \frac{7}{32}}{2} = \frac{3}{32}.
\]

107. \[ \lim_{{t \to \infty}} \frac{\ln(t + 2)}{\ln 2} \]

**SOLUTION** The limit is an indeterminate form of type \( \frac{\infty}{\infty} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{t \to \infty}} \frac{\ln(t + 2)}{\ln 2} = \lim_{{t \to \infty}} \frac{\frac{1}{t + 2}}{\frac{1}{\ln 2}} = \lim_{{t \to \infty}} \frac{t \ln 2}{t + 2} = \lim_{{t \to \infty}} \frac{2}{1} = \ln 2.
\]

108. \[ \lim_{{x \to 0}} \left( \frac{e^x}{e^x - 1} - \frac{1}{x} \right) \]

**SOLUTION** First rewrite the function as a quotient:

\[
\frac{e^x}{e^x - 1} - \frac{1}{x} = \frac{xe^x - e^x + 1}{x(e^x - 1)}.
\]

The limit is now an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{x \to 0}} \left( \frac{e^x}{e^x - 1} - \frac{1}{x} \right) = \lim_{{x \to 0}} \frac{xe^x + e^x - e^x}{xe^x + e^x - 1} = \lim_{{x \to 0}} \frac{xe^x}{xe^x + e^x} = \frac{1}{1 + 1} = \frac{1}{2}.
\]

109. \[ \lim_{{y \to 0}} \frac{\sin^{-1} y - y}{y^3} \]

**SOLUTION** The limit is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{y \to 0}} \frac{\sin^{-1} y - y}{y^3} = \lim_{{y \to 0}} \frac{1}{\sqrt{1 - y^2}^3} = \lim_{{y \to 0}} \frac{y(1 - y^2)^{-3/2}}{6y} = \lim_{{y \to 0}} \frac{(1 - y^2)^{-3/2}}{6} = \frac{1}{6}.
\]

110. \[ \lim_{{x \to 1}} \frac{\sqrt{1 - x^2}}{\cos^{-1} x} \]

**SOLUTION** The limit is an indeterminate form \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{x \to 1}} \frac{\sqrt{1 - x^2}}{\cos^{-1} x} = \lim_{{x \to 1}} \frac{-x}{\sqrt{1 - x^2}} = \lim_{{x \to 1}} x = 1.
\]

111. \[ \lim_{{x \to 0}} \frac{\sinh(x^2)}{\cosh x - 1} \]

**SOLUTION** The limit is an indeterminate form \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{{x \to 0}} \frac{\sinh(x^2)}{\cosh x - 1} = \lim_{{x \to 0}} \frac{2x \cosh(x^2)}{\sinh x} = \lim_{{x \to 0}} \frac{2 \cosh(x^2) + 4x^2 \sinh(x^2)}{\cosh x} = \frac{2 + 0}{1} = 2.
\]
112. \( \lim_{x \to 0} \frac{\tanh x - \sinh x}{\sin x - x} \)

**SOLUTION**  The limit is an indeterminate form of type \( \frac{0}{0} \); hence, we may apply L'Hôpital's Rule. We find

\[
\lim_{x \to 0} \frac{\tanh x - \sinh x}{\sin x - x} = \lim_{x \to 0} \frac{\text{sech}^2 x - \cosh x}{\cos x - 1} = \lim_{x \to 0} \frac{2 \text{sech} x (-\text{sech} x \tanh x) - \sinh x}{-\sin x} = \lim_{x \to 0} \frac{2 \text{sech}^2 x \tanh x + \sinh x}{\sin x} \frac{1}{\sin x} = \lim_{x \to 0} \frac{-4 \text{sech}^2 x \tanh^2 x + 2 \text{sech}^4 x + \cosh x}{\cos x} = -4 \cdot 1 \cdot 0 + 2 \cdot 1 + 1 = 3.
\]

113. Let \( f(x) = e^{-Ax^2/2} \), where \( A > 0 \). Given any \( n \) numbers \( a_1, a_2, \ldots, a_n \), set

\[ \Phi(x) = f(x - a_1) f(x - a_2) \cdots f(x - a_n) \]

(a) Assume \( n = 2 \) and prove that \( \Phi(x) \) attains its maximum value at the average \( x = \frac{1}{2}(a_1 + a_2) \). *Hint:* Calculate \( \Phi'(x) \) using logarithmic differentiation.

(b) Show that for any \( n \), \( \Phi(x) \) attains its maximum value at \( x = \frac{1}{2}(a_1 + a_2 + \cdots + a_n) \). This fact is related to the role of \( f(x) \) (whose graph is a bell-shaped curve) in statistics.

**SOLUTION**

(a) For \( n = 2 \) we have,

\[ (x) = f(x - a_1) f(x - a_2) = e^{-\frac{1}{2}(x-a_1)^2} \cdot e^{-\frac{1}{2}(x-a_2)^2} = e^{-\frac{1}{2}((x-a_1)^2+(x-a_2)^2)}. \]

Since \( e^{-\frac{1}{2}y} \) is a decreasing function of \( y \), it attains its maximum value where \( y \) is minimum. Therefore, we must find the minimum value of

\[ y = (x - a_1)^2 + (x - a_2)^2 = 2x^2 - 2(a_1 + a_2)x + a_1^2 + a_2^2. \]

Now, \( y' = 4x - 2(a_1 + a_2) = 0 \) when

\[ x = \frac{a_1 + a_2}{2}. \]

We conclude that \( (x) \) attains a maximum value at this point.

(b) We have

\[ (x) = e^{-\frac{1}{2}(x-a_1)^2} \cdot e^{-\frac{1}{2}(x-a_2)^2} \cdots e^{-\frac{1}{2}(x-a_n)^2} = e^{-\frac{1}{2}((x-a_1)^2+\cdots+(x-a_n)^2)}. \]

Since the function \( e^{-\frac{1}{2}y} \) is a decreasing function of \( y \), it attains a maximum value where \( y \) is minimum. Therefore we must minimize the function

\[ y = (x - a_1)^2 + (x - a_2)^2 + \cdots + (x - a_n)^2. \]

We find the critical points by solving:

\[ y' = 2(x - a_1) + 2(x - a_2) + \cdots + 2(x - a_n) = 0 \]

\[ 2nx = 2(a_1 + a_2 + \cdots + a_n) \]

\[ x = \frac{a_1 + \cdots + a_n}{n}. \]

We verify that this point corresponds the minimum value of \( y \) by examining the sign of \( y'' \) at this point: \( y'' = 2n > 0 \). We conclude that \( y \) attains a minimum value at the point \( x = \frac{a_1 + \cdots + a_n}{n} \), hence \( (x) \) attains a maximum value at this point.