CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function $f(x)$ whose graph is shown in Figure 1.

![Figure 1](image)

1. Estimate $L_4$ and $M_4$ on $[0, 4]$.

**Solution** With $n = 4$ and an interval of $[0, 4]$, $\Delta x = \frac{4-0}{4} = 1$. Then,

$$L_4 = \Delta x (f(0) + f(1) + f(2) + f(3)) = 1 \left( \frac{1}{4} + 1 + \frac{5}{2} + 2 \right) = \frac{23}{4}$$

and

$$M_4 = \Delta x \left( f \left( \frac{1}{2} \right) + f \left( \frac{3}{2} \right) + f \left( \frac{5}{2} \right) + f \left( \frac{7}{2} \right) \right) = 1 \left( \frac{1}{2} + 2 + \frac{9}{4} + \frac{9}{4} \right) = 7.$$ 

2. Estimate $R_4$, $L_4$, and $M_4$ on $[1, 3]$.

**Solution** With $n = 4$ and an interval of $[1, 3]$, $\Delta x = \frac{3-1}{4} = \frac{1}{2}$. Then,

$$R_4 = \Delta x \left( f \left( \frac{3}{2} \right) + f(2) + f \left( \frac{5}{2} \right) + f(3) \right) = \frac{1}{2} \left( 2 + \frac{5}{2} + \frac{9}{4} + 2 \right) = \frac{35}{8};$$

$$L_4 = \Delta x \left( f(1) + f \left( \frac{3}{2} \right) + f(2) + f \left( \frac{5}{2} \right) \right) = \frac{1}{2} \left( 1 + 2 + \frac{5}{2} + \frac{9}{4} \right) = \frac{31}{8};$$

and

$$M_4 = \Delta x \left( f \left( \frac{5}{4} \right) + f \left( \frac{7}{4} \right) + f \left( \frac{9}{4} \right) + f \left( \frac{11}{4} \right) \right) = \frac{1}{2} \left( \frac{3}{2} + \frac{9}{4} + \frac{5}{2} + \frac{17}{8} \right) = \frac{67}{16}.$$ 

3. Find an interval $[a, b]$ on which $R_4$ is larger than $\int_a^b f(x) \, dx$. Do the same for $L_4$.

**Solution** In general, $R_N$ is larger than $\int_a^b f(x) \, dx$ on any interval $[a, b]$ over which $f(x)$ is increasing. Given the graph of $f(x)$, we may take $[a, b] = [0, 2]$. In order for $L_4$ to be larger than $\int_a^b f(x) \, dx$, $f(x)$ must be decreasing over the interval $[a, b]$. We may therefore take $[a, b] = [2, 3]$.

4. Justify $\frac{3}{2} \leq \int_1^2 f(x) \, dx \leq \frac{9}{4}$.

**Solution** Because $f(x)$ is increasing on $[1, 2]$, we know that

$$L_N \leq \int_1^2 f(x) \, dx \leq R_N$$

for any $N$. Now,

$$L_2 = \frac{1}{2} (1 + 2) = \frac{3}{2} \quad \text{and} \quad R_2 = \frac{1}{2} \left( 2 + \frac{5}{2} \right) = \frac{9}{4},$$

so

$$\frac{3}{2} \leq \int_1^2 f(x) \, dx \leq \frac{9}{4}.$$
In Exercises 5–8, let $f(x) = x^2 + 3x$.

5. Calculate $R_6$, $M_6$, and $L_6$ for $f(x)$ on the interval $[2, 5]$. Sketch the graph of $f(x)$ and the corresponding rectangles for each approximation.

**SOLUTION** Let $f(x) = x^2 + 3x$. A uniform partition of $[2, 5]$ with $N = 6$ subintervals has

$$\Delta x = \frac{5 - 2}{6} = \frac{1}{2}, \quad x_j = a + j\Delta x = 2 + \frac{j}{2},$$

and

$$x_j^* = a + \left(j - \frac{1}{2}\right)\Delta x = \frac{7}{4} + \frac{j}{2}.$$

Now,

$$R_6 = \Delta x \sum_{j=1}^{6} f(x_j) = \frac{1}{2} \left(f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right) + f(5)\right)$$

$$= \frac{1}{2} \left(\frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4} + 40\right) = \frac{625}{8}.$$

The rectangles corresponding to this approximation are shown below.

Next,

$$M_6 = \Delta x \sum_{j=1}^{6} f(x_j^*) = \frac{1}{2} \left(f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right)$$

$$= \frac{1}{2} \left(\frac{189}{16} + \frac{253}{16} + \frac{325}{16} + \frac{405}{16} + \frac{493}{16} + \frac{589}{16}\right) = \frac{2254}{32} = \frac{1127}{16}.$$

The rectangles corresponding to this approximation are shown below.

Finally,

$$L_6 = \Delta x \sum_{j=0}^{5} f(x_j) = \frac{1}{2} \left(f(2) + f\left(\frac{5}{2}\right) + f(3) + f\left(\frac{7}{2}\right) + f(4) + f\left(\frac{9}{2}\right)\right)$$

$$= \frac{1}{2} \left(10 + \frac{55}{4} + 18 + \frac{91}{4} + 28 + \frac{135}{4}\right) = \frac{505}{8}.$$

The rectangles corresponding to this approximation are shown below.
6. Use FTC I to evaluate \( A(x) = \int_{-2}^{x} f(t) \, dt \).

**SOLUTION**  
Let \( f(x) = x^2 + 3x \). Then
\[
A(x) = \int_{-2}^{x} (t^2 + 3t) \, dt = \left( \frac{1}{3}t^3 + \frac{3}{2}t^2 \right) \bigg|_{-2}^{x} = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \left( -\frac{8}{3} + 6 \right) = \frac{1}{3}x^3 + \frac{3}{2}x^2 - \frac{10}{3}.
\]

7. Find a formula for \( R_N \) for \( f(x) \) on \([2, 5] \) and compute \( \int_{2}^{5} f(x) \, dx \) by taking the limit.

**SOLUTION**  
Let \( f(x) = x^2 + 3x \) on the interval \([2, 5] \). Then \( \Delta x = \frac{5 - 2}{N} = \frac{3}{N} \) and \( a = 2 \). Hence,
\[
R_N = \Delta x \sum_{j=1}^{N} f(2 + j\Delta x) = \frac{3}{N} \sum_{j=1}^{N} \left( \left( 2 + \frac{3j}{N} \right)^2 + 3 \left( 2 + \frac{3j}{N} \right) \right) = \frac{3}{N} \sum_{j=1}^{N} \left( 10 + \frac{21j}{N} + \frac{9j^2}{N^2} \right)
\]
\[
= 30 + \frac{63}{N^2} \sum_{j=1}^{N} j + \frac{27}{N^3} \sum_{j=1}^{N} j^2
\]
\[
= 30 + \frac{63}{N^2} \left( \frac{N^2}{2} + \frac{N}{2} \right) + \frac{27}{N^3} \left( \frac{N^3}{3} + \frac{N^2}{2} + \frac{N}{6} \right)
\]
\[
= \frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2}
\]
and
\[
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left( \frac{141}{2} + \frac{45}{N} + \frac{9}{2N^2} \right) = \frac{141}{2}.
\]

8. Find a formula for \( L_N \) for \( f(x) \) on \([0, 2] \) and compute \( \int_{0}^{2} f(x) \, dx \) by taking the limit.

**SOLUTION**  
Let \( f(x) = x^2 + 3x \) and \( N \) be a positive integer. Then
\[
\Delta x = \frac{2 - 0}{N} = \frac{2}{N}
\]
and
\[
x_j = a + j\Delta x = 0 + \frac{2j}{N} = \frac{2j}{N}
\]
for \( 0 \leq j \leq N \). Thus,
\[
L_N = \Delta x \sum_{j=0}^{N-1} f(x_j) = \frac{2}{N} \sum_{j=0}^{N-1} \left( \frac{4j^2}{N^2} + \frac{6j}{N} \right) = \frac{8}{N^3} \sum_{j=0}^{N-1} j^2 + \frac{12}{N^2} \sum_{j=0}^{N-1} j
\]
\[
= \frac{4(N - 1)(2N - 1)}{3N^2} + \frac{6(N - 1)}{N} = \frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2}.
\]
Finally,
\[
\int_{0}^{2} f(x) \, dx = \lim_{N \to \infty} \left( \frac{26}{3} - \frac{10}{N} + \frac{4}{3N^2} \right) = \frac{26}{3}.
\]

9. Calculate \( R_5, M_5, \) and \( L_5 \) for \( f(x) = (x^2 + 1)^{-1} \) on the interval \([0, 1] \).

**SOLUTION**  
Let \( f(x) = (x^2 + 1)^{-1} \). A uniform partition of \([0, 1] \) with \( N = 5 \) subintervals has
\[
\Delta x = \frac{1 - 0}{5} = \frac{1}{5}, \quad x_j = a + j\Delta x = \frac{j}{5},
\]
and
\[
x_j^* = a + \left( j - \frac{1}{2} \right) \Delta x = \frac{2j - 1}{10}.
\]
Now,
\[
R_5 = \Delta x \sum_{j=1}^{5} f(x_j) = \frac{1}{5} \left( f \left( \frac{1}{5} \right) + f \left( \frac{2}{5} \right) + f \left( \frac{3}{5} \right) + f \left( \frac{4}{5} \right) + f(1) \right)
\]
\[
\frac{1}{5} \left( \frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41} + \frac{1}{2} \right) \approx 0.733732.
\]

Next,
\[
M_5 = \Delta x \sum_{j=1}^{5} f(x_j) = \frac{1}{5} \left( f \left( \frac{1}{10} \right) + f \left( \frac{3}{10} \right) + f \left( \frac{1}{2} \right) + f \left( \frac{7}{10} \right) + f \left( \frac{9}{10} \right) \right)
\]
\[
= \frac{1}{5} \left( \frac{100}{101} + \frac{100}{109} + \frac{4}{5} + \frac{100}{149} + \frac{100}{181} \right) \approx 0.786231.
\]

Finally,
\[
L_5 = \Delta x \sum_{j=0}^{4} f(x_j) = \frac{1}{5} \left( f(0) + f \left( \frac{1}{5} \right) + f \left( \frac{2}{5} \right) + f \left( \frac{3}{5} \right) + f \left( \frac{4}{5} \right) \right)
\]
\[
= \frac{1}{5} \left( 1 + \frac{25}{26} + \frac{25}{29} + \frac{25}{34} + \frac{25}{41} \right) \approx 0.833732.
\]

10. Let \( R_N \) be the \( N \)th right-endpoint approximation for \( f(x) = x^3 \) on \([0, 4]\) (Figure 2).

(a) Prove that \( R_N = \frac{64(N + 1)^2}{N^2} \).

(b) Prove that the area of the region within the right-endpoint rectangles above the graph is equal to
\[
\frac{64(2N + 1)}{N^2}
\]

\[\text{FIGURE 2 Approximation } R_N \text{ for } f(x) = x^3 \text{ on } [0, 4].\]

**SOLUTION**

(a) Let \( f(x) = x^3 \) and \( N \) be a positive integer. Then
\[
\Delta x = \frac{4 - 0}{N} = \frac{4}{N} \quad \text{and} \quad x_j = a + j\Delta x = 0 + \frac{4j}{N} = \frac{4j}{N}
\]
for \( 0 \leq j \leq N \). Thus,
\[
R_N = \Delta x \sum_{j=1}^{N} f(x_j) = \frac{4}{N} \sum_{j=1}^{N} \frac{64j^3}{N^3} = \frac{256}{N^4} \sum_{j=1}^{N} j^3 = \frac{256}{N^4} \frac{N^2(N + 1)^2}{4} = \frac{64(N + 1)^2}{N^2}.
\]

(b) The area between the graph of \( y = x^3 \) and the \( x \)-axis over \([0, 4]\) is
\[
\int_{0}^{4} x^3 \, dx = \left. \frac{1}{4} x^4 \right|_{0}^{4} = 64.
\]

The area of the region below the right-endpoint rectangles and above the graph is therefore
\[
\frac{64(N + 1)^2}{N^2} - 64 = \frac{64(2N + 1)}{N^2}.
\]

11. Which approximation to the area is represented by the shaded rectangles in Figure 3? Compute \( R_5 \) and \( L_5 \).

\[\text{FIGURE 3}\]
SOLUTION There are five rectangles and the height of each is given by the function value at the right endpoint of the subinterval. Thus, the area represented by the shaded rectangles is \( R_5 \).

From the figure, we see that \( \Delta x = 1 \). Then

\[
R_5 = 1(30 + 18 + 6 + 6 + 30) = 90 \quad \text{and} \quad L_5 = 1(30 + 30 + 18 + 6 + 6) = 90.
\]

12. Calculate any two Riemann sums for \( f(x) = x^2 \) on the interval \([2, 5]\), but choose partitions with at least five subintervals of unequal widths and intermediate points that are neither endpoints nor midpoints.

SOLUTION Let \( f(x) = x^2 \). Riemann sums will, of course, vary. Here are two possibilities. Take \( N = 5 \),

\[
P = \{x_0 = 2, x_1 = 2.7, x_2 = 3.1, x_3 = 3.6, x_4 = 4.2, x_5 = 5\}
\]

and

\[
C = \{c_1 = 2.5, c_2 = 3, c_3 = 3.5, c_4 = 4, c_5 = 4.5\}.
\]

Then,

\[
R(f, P, C) = \sum_{j=1}^{5} \Delta x_j f(c_j) = 0.7(6.25) + 0.4(9) + 0.5(12.25) + 0.6(16) + 0.8(20.25) = 39.9.
\]

Alternatively, take \( N = 6 \),

\[
P = \{x_0 = 2, x_1 = 2.5, x_2 = 3.5, x_3 = 4, x_4 = 4.25, x_5 = 4.75, x_6 = 5\}
\]

and

\[
C = \{c_1 = 2.1, c_2 = 3, c_3 = 3.7, c_4 = 4.2, c_5 = 4.5, c_6 = 4.8\}.
\]

Then,

\[
R(f, P, C) = \sum_{j=1}^{6} \Delta x_j f(c_j)
\]

\[
= 0.5(4.41) + 1(9) + 0.5(13.69) + 0.25(17.64) + 0.5(20.25) + 0.25(23.04) = 38.345.
\]

In Exercises 13–16, express the limit as an integral (or multiple of an integral) and evaluate.

13. \( \lim_{N \to \infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) \)

SOLUTION Let \( f(x) = \sin x \) and \( N \) be a positive integer. A uniform partition of the interval \([\pi/3, \pi/2]\) with \( N \) subintervals has

\[
\Delta x = \frac{\pi}{6N} \quad \text{and} \quad x_j = \frac{\pi}{3} + \frac{\pi j}{6N}
\]

for \( 0 \leq j \leq N \). Then

\[
\frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;
\]

consequently,

\[
\lim_{N \to \infty} \frac{\pi}{6N} \sum_{j=1}^{N} \sin\left(\frac{\pi}{3} + \frac{\pi j}{6N}\right) = \int_{\pi/3}^{\pi/2} \sin x \, dx = -\cos x \bigg|_{\pi/3}^{\pi/2} = 0 + \frac{1}{2} = \frac{1}{2}.
\]

14. \( \lim_{N \to \infty} \frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) \)

SOLUTION Let \( f(x) = x \) and \( N \) be a positive integer. A uniform partition of the interval \([10, 13]\) with \( N \) subintervals has

\[
\Delta x = \frac{3}{N} \quad \text{and} \quad x_j = 10 + \frac{3j}{N}
\]

for \( 0 \leq j \leq N \). Then

\[
\frac{3}{N} \sum_{k=0}^{N-1} \left(10 + \frac{3k}{N}\right) = \Delta x \sum_{j=0}^{N-1} f(x_j) = L_N;
\]
consequently,
\[
\lim_{N \to \infty} \frac{3}{N} \sum_{k=0}^{N-1} \left( 10 + \frac{3k}{N} \right) = \int_{0}^{13} x \, dx = \frac{1}{2} x^2 \bigg|_{10}^{13} = \frac{169}{2} - \frac{100}{2} = 69.
\]

15. \[ \lim_{N \to \infty} \frac{5}{N} \sum_{j=1}^{N} \sqrt{4 + \frac{5j}{N}} \]

**SOLUTION** Let \( f(x) = \sqrt{x} \) and \( N \) be a positive integer. A uniform partition of the interval \([4, 9]\) with \( N \) subintervals has
\[
\Delta x = \frac{5}{N} \quad \text{and} \quad x_j = 4 + \frac{5j}{N}
\]
for \( 0 \leq j \leq N \). Then
\[
\frac{5}{N} \sum_{j=1}^{N} \sqrt{4 + \frac{5j}{N}} = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;
\]
consequently,
\[
\lim_{N \to \infty} \frac{5}{N} \sum_{j=1}^{N} \sqrt{4 + \frac{5j}{N}} = \int_{4}^{9} \sqrt{x} \, dx = \frac{2}{3} x^{3/2} \bigg|_{4}^{9} = \frac{54}{3} - \frac{16}{3} = \frac{38}{3}.
\]

16. \[ \lim_{N \to \infty} \frac{1^k + 2^k + \ldots + N^k}{N^{k+1}} \quad (k > 0) \]

**SOLUTION** Observe that
\[
\frac{1^k + 2^k + \ldots + N^k}{N^{k+1}} = \frac{1}{N} \left[ \left( \frac{1}{N} \right)^k + \left( \frac{2}{N} \right)^k + \left( \frac{3}{N} \right)^k + \ldots + \left( \frac{N}{N} \right)^k \right] = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^k.
\]
Now, let \( f(x) = x^k \) and \( N \) be a positive integer. A uniform partition of the interval \([0, 1]\) with \( N \) subintervals has
\[
\Delta x = \frac{1}{N} \quad \text{and} \quad x_j = \frac{j}{N}
\]
for \( 0 \leq j \leq N \). Then
\[
\frac{1}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^k = \Delta x \sum_{j=1}^{N} f(x_j) = R_N;
\]
consequently,
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \left( \frac{j}{N} \right)^k = \int_{0}^{1} x^k \, dx = \frac{1}{k+1} x^{k+1} \bigg|_{0}^{1} = \frac{1}{k+1}.
\]

*In Exercises 17–20, use the given substitution to evaluate the integral.*

17. \[ \int_{0}^{2} \frac{dt}{4t + 12}, \quad u = 4t + 12 \]

**SOLUTION** Let \( u = 4t + 12 \). Then \( du = 4 \, dt \), and the new limits of integration are \( u = 12 \) and \( u = 20 \). Thus,
\[
\int_{0}^{2} \frac{dt}{4t + 12} = \frac{1}{4} \int_{12}^{20} \frac{du}{u} = \frac{1}{4} \ln u \bigg|_{12}^{20} = \frac{1}{4} \ln 20 - \frac{1}{4} \ln 12 = \frac{1}{4} \ln \frac{20}{12} = \frac{1}{4} \ln \frac{5}{3}.
\]

18. \[ \int \frac{(x^2 + 1) \, dx}{(x^3 + 3x)^k}, \quad u = x^3 + 3x \]

**SOLUTION** Let \( u = x^3 + 3x \). Then \( du = (3x^2 + 3) \, dx = 3(x^2 + 1) \, dx \) and
\[
\int \frac{(x^2 + 1) \, dx}{(x^3 + 3x)^k} = \frac{1}{3} \int u^{-4} \, du = -\frac{1}{9} u^{-3} + C = -\frac{1}{9} (x^3 + 3x)^{-3} + C.
\]
19. \( \int_{0}^{\pi/6} \sin x \cos^4 x \, dx \), \( u = \cos x \)

**SOLUTION** Let \( u = \cos x \). Then \( du = -\sin x \, dx \) and the new limits of integration are \( u = 1 \) and \( u = \sqrt{3}/2 \). Thus,

\[
\int_{0}^{\pi/6} \sin x \cos^4 x \, dx = - \int_{1}^{\sqrt{3}/2} u^4 \, du
\]

\[
= - \left. \frac{1}{5}u^5 \right|_{1}^{\sqrt{3}/2}
\]

\[
= \frac{1}{5} \left( 1 - \frac{9\sqrt{3}}{32} \right).
\]

20. \( \int \sec^2(2\theta) \tan(2\theta) \, d\theta \), \( u = \tan(2\theta) \)

**SOLUTION** Let \( u = \tan(2\theta) \). Then \( du = 2\sec^2(2\theta) \, d\theta \) and

\[
\int \sec^2(2\theta) \tan(2\theta) \, d\theta = \frac{1}{2} \int u \, du = \frac{1}{4}u^2 + C = \frac{1}{4} \tan^2(2\theta) + C.
\]

*In Exercises 21–70, evaluate the integral.*

21. \( \int (20x^4 - 9x^3 - 2x) \, dx \)

**SOLUTION**

\[
\int (20x^4 - 9x^3 - 2x) \, dx = 4x^5 - \frac{9}{4}x^4 - x^2 + C.
\]

22. \( \int (12x^3 - 3x^2) \, dx \)

**SOLUTION**

\[
\int (12x^3 - 3x^2) \, dx = (3x^4 - x^3) \bigg|_{0}^{2} = (48 - 8) - 0 = 40.
\]

23. \( \int (2x^2 - 3x)^2 \, dx \)

**SOLUTION**

\[
\int (2x^2 - 3x)^2 \, dx = \int (4x^4 - 12x^3 + 9x^2) \, dx = \frac{4}{5}x^5 - 3x^4 + 3x^3 + C.
\]

24. \( \int (x^{7/3} - 2x^{1/4}) \, dx \)

**SOLUTION**

\[
\int (x^{7/3} - 2x^{1/4}) \, dx = \left( \frac{3}{10}x^{10/3} - \frac{8}{5}x^{5/4} \right) \bigg|_{0}^{1} = \frac{3}{10} - \frac{8}{5} = -\frac{13}{10}.
\]

25. \( \int \frac{x^5 + 3x^4}{x^2} \, dx \)

**SOLUTION**

\[
\int \frac{x^5 + 3x^4}{x^2} \, dx = \int (x^3 + 3x^2) \, dx = \frac{1}{4}x^4 + x^3 + C.
\]

26. \( \int_{1}^{3} r^{-4} \, dr \)

**SOLUTION**

\[
\int_{1}^{3} r^{-4} \, dr = -\frac{1}{3}r^{-3} \bigg|_{1}^{3} = -\frac{1}{3} \left( \frac{1}{27} - 1 \right) = \frac{26}{81}.
\]

27. \( \int_{-3}^{3} |x^2 - 4| \, dx \)

**SOLUTION**

\[
\int_{-3}^{3} |x^2 - 4| \, dx = \int_{-3}^{2} (x^2 - 4) \, dx + \int_{2}^{4} (4 - x^2) \, dx + \int_{4}^{3} (x^2 - 4) \, dx
\]

\[
= \left( \frac{1}{3}x^3 - 4x \right) \bigg|_{-3}^{2} + \left( 4x - \frac{1}{3}x^3 \right) \bigg|_{2}^{4} + \left( \frac{1}{3}x^3 - 4x \right) \bigg|_{4}^{3}
\]

\[
= \left( \frac{16}{3} - 3 \right) + \left( \frac{16}{3} + \frac{16}{3} \right) + \left( -3 + \frac{16}{3} \right)
\]

\[
= \frac{46}{3}.
\]
28. \[ \int_{-2}^{4} |(x - 1)(x - 3)| \, dx \]

**SOLUTION**

\[
\int_{-2}^{4} (x - 1)(x - 3) \, dx = \int_{-2}^{1} (x^2 - 4x + 3) \, dx + \int_{1}^{3} (-x^2 + 4x - 3) \, dx + \int_{3}^{4} (x^2 - 4x + 3) \, dx \\
= \left( \frac{1}{3}x^3 - 2x^2 + 3x \right|_{-2}^{1} + \left( -\frac{1}{3}x^3 + 2x^2 - 3x \right|_{1}^{3} + \left( \frac{1}{3}x^3 - 2x^2 + 3x \right|_{3}^{4} \\
= \frac{4}{3} - \frac{50}{3} + 0 - \left( -\frac{4}{3} \right) + \frac{4}{3} - 0 \\
= \frac{62}{3}.
\]

29. \[ \int_{1}^{3} [t] \, dt \]

**SOLUTION**

\[
\int_{1}^{3} [t] \, dt = \int_{1}^{2} [t] \, dt + \int_{2}^{3} [t] \, dt = \int_{1}^{2} dt + \int_{2}^{3} 2 \, dt = 1 + 2[3] = 2 - 1 + 6 - 4 = 3.
\]

30. \[ \int_{0}^{2} (t - [t])^2 \, dt \]

**SOLUTION**

\[
\int_{0}^{2} (t - [t])^2 \, dt = \int_{0}^{1} t^2 \, dt + \int_{1}^{2} (t - 1)^2 \, dt \\
= \left. \frac{1}{3}t^3 \right|_{0}^{1} + \left. \frac{1}{3}(t - 1)^3 \right|_{1}^{2} \\
= \frac{1}{3} + \frac{2}{3} = \frac{2}{3}.
\]

31. \[ \int (10t - 7)^{14} \, dt \]

**SOLUTION** Let \( u = 10t - 7 \). Then \( du = 10 \, dt \) and

\[
\int (10t - 7)^{14} \, dt = \frac{1}{10} \int u^{14} \, du = \frac{1}{150} u^{15} + C = \frac{1}{150} (10t - 7)^{15} + C.
\]

32. \[ \int_{2}^{3} \sqrt{7y - 5} \, dy \]

**SOLUTION** Let \( u = 7y - 5 \). Then \( du = 7 \, dy \) and when \( y = 2, u = 9 \) and when \( y = 3, u = 16 \). Finally,

\[
\int_{2}^{3} \sqrt{7y - 5} \, dy = \frac{1}{7} \int_{9}^{16} u^{1/2} \, du = \frac{1}{7} \cdot \frac{2}{3} u^{3/2} \bigg|_{9}^{16} = \frac{2}{21} (64 - 27) = \frac{74}{21}.
\]

33. \[ \int \frac{(2x^3 + 3x)}{(3x^4 + 9x^2)^3} \, dx \]

**SOLUTION** Let \( u = 3x^4 + 9x^2 \). Then \( du = (12x^3 + 18x) \, dx \) and

\[
\int \frac{(2x^3 + 3x)}{(3x^4 + 9x^2)^3} \, dx = \frac{1}{6} \int u^{-5} \, du = -\frac{1}{24} u^{-4} + C = -\frac{1}{24} (3x^4 + 9x^2)^{-4} + C.
\]

34. \[ \int_{-3}^{1} \frac{x \, dx}{(x^2 + 5)^2} \]

**SOLUTION** Let \( u = x^2 + 5 \). Then \( du = 2x \, dx \) and

\[
\int_{-3}^{1} \frac{x \, dx}{(x^2 + 5)^2} = \frac{1}{2} \int_{14}^{6} u^{-2} \, du = -\frac{1}{2} u^{-1} \bigg|_{14}^{6} \\
= -\frac{1}{2} \left( \frac{1}{6} - \frac{1}{14} \right) = -\frac{1}{21}.
\]
35. \[\int_{0}^{5} 15x \sqrt{x + 4} \, dx\]

**SOLUTION** Let \(u = x + 4\). Then \(x = u - 4\), \(du = dx\) and the new limits of integration are \(u = 4\) and \(u = 9\). Thus,

\[
\int_{0}^{5} 15x \sqrt{x + 4} \, dx = \int_{4}^{9} 15(u - 4) \sqrt{u} \, du
\]

\[
= 15 \int_{4}^{9} (u^{3/2} - 4u^{1/2}) \, du
\]

\[
= 15 \left[ \left( \frac{2}{3}u^{5/2} - \frac{8}{3}u^{3/2} \right) \right]_{4}^{9}
\]

\[
= 15 \left( \frac{486}{5} - 72 - \left( \frac{64}{5} - \frac{64}{3} \right) \right)
\]

\[
= 506.
\]

36. \[\int t^{2} \sqrt{t + 8} \, dt\]

**SOLUTION** Let \(u = t + 8\). Then \(du = dt\), \(t = u - 8\), and

\[
\int t^{2} \sqrt{t + 8} \, dt = \int (u - 8)^{2} \sqrt{u} \, du
\]

\[
= \int (u^{3/2} - 16u^{1/2} + 64u^{-1/2}) \, du
\]

\[
= \frac{2}{7}u^{7/2} - \frac{32}{5}u^{5/2} + \frac{128}{3}u^{3/2} + C
\]

\[
= \frac{2}{7}(t + 8)^{7/2} - \frac{32}{5}(t + 8)^{5/2} + \frac{128}{3}(t + 8)^{3/2} + C.
\]

37. \[\int_{0}^{1} \cos \left( \frac{\pi}{3}(t + 2) \right) \, dt\]

**SOLUTION** \[\int_{0}^{1} \cos \left( \frac{\pi}{3}(t + 2) \right) \, dt = \frac{3}{\pi} \sin \left( \frac{\pi}{3}(t + 2) \right) \bigg|_{0}^{1} = -\frac{3\sqrt{3}}{2\pi}.
\]

38. \[\int_{\pi/2}^{\pi} \sin \left( \frac{5\theta - \pi}{6} \right) \, d\theta\]

**SOLUTION** Let

\[
u = \frac{5\theta - \pi}{6}\]

so that \(du = \frac{5}{6}d\theta\).

Then

\[
\int_{\pi/2}^{\pi} \sin \left( \frac{5\theta - \pi}{6} \right) \, d\theta = \frac{6}{5} \int_{\pi/4}^{2\pi/3} \sin u \, du
\]

\[
= -\frac{6}{5} \cos u \bigg|_{\pi/4}^{2\pi/3}
\]

\[
= -\frac{6}{5} \left( \frac{1}{2} - \frac{\sqrt{2}}{2} \right) = \frac{3}{5}(1 + \sqrt{2}).
\]

39. \[\int t^{2} \sec^{2}(9t^{3} + 1) \, dt\]

**SOLUTION** Let \(u = 9t^{3} + 1\). Then \(du = 27t^{2} \, dt\) and

\[
\int t^{2} \sec^{2}(9t^{3} + 1) \, dt = \frac{1}{27} \int \sec^{2} u \, du = \frac{1}{27} \tan u + C = \frac{1}{27} \tan(9t^{3} + 1) + C.
\]

40. \[\int \sin^{2}(3\theta) \cos(3\theta) \, d\theta\]

**SOLUTION** Let \(u = \sin(3\theta)\). Then \(du = 3\cos(3\theta) \, d\theta\) and

\[
\int \sin^{2}(3\theta) \cos(3\theta) \, d\theta = \frac{1}{3} \int u^{2} \, du = \frac{1}{9}u^{3} + C = \frac{1}{9} \sin^{3}(3\theta) + C.
\]
41. \( \int \csc^2(9 - 2\theta) \, d\theta \)

**SOLUTION** Let \( u = 9 - 2\theta \). Then \( du = -2 \, d\theta \) and

\[
\int \csc^2(9 - 2\theta) \, d\theta = -\frac{1}{2} \int \csc^2 u \, du = -\frac{1}{2} \cot u + C = -\frac{1}{2} \cot(9 - 2\theta) + C.
\]

42. \( \int \sin \theta \sqrt{4 - \cos \theta} \, d\theta \)

**SOLUTION** Let \( u = 4 - \cos \theta \). Then \( du = \sin \theta \, d\theta \) and

\[
\int \sin \theta \sqrt{4 - \cos \theta} \, d\theta = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (4 - \cos \theta)^{3/2} + C.
\]

43. \( \int_0^{\pi/3} \frac{\sin \theta}{\cos^2 \theta} \, d\theta \)

**SOLUTION** Let \( u = \cos \theta \). Then \( du = -\sin \theta \, d\theta \) and when \( \theta = 0, u = 1 \) and when \( \theta = \frac{\pi}{3}, u = \frac{1}{2} \). Finally,

\[
\int_0^{\pi/3} \frac{\sin \theta}{\cos^2 \theta} \, d\theta = -\left. \int_1^{1/2} u^{-2/3} \, du \right|_{1}^{1/2} = -3u^{1/3} \bigg|_{1}^{1/2} = -3(2^{-1/3} - 1) = 3 - \frac{3\sqrt{3}}{2}.
\]

44. \( \int \sec^2 t \, dt \)

**SOLUTION** Let \( u = \tan t - 1 \). Then \( du = \sec^2 t \, dt \) and

\[
\int \frac{\sec^2 t \, dt}{(\tan t - 1)^2} = \int u^{-2} \, du = -u^{-1} + C = -\frac{1}{\tan t - 1} + C.
\]

45. \( \int e^{9-2x} \, dx \)

**SOLUTION** Let \( u = 9 - 2x \). Then \( du = -2 \, dx \), and

\[
\int e^{9-2x} \, dx = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = \frac{1}{2} e^{9-2x} + C.
\]

46. \( \int_1^3 e^{4x-3} \, dx \)

**SOLUTION**

\[
\int_1^3 e^{4x-3} \, dx = \frac{1}{4} e^{4x-3} \bigg|_1^3 = \frac{1}{4} (e^9 - e).
\]

47. \( \int x^2 e^{x^3} \, dx \)

**SOLUTION** Let \( u = x^3 \). Then \( du = 3x^2 \, dx \), and

\[
\int x^2 e^{x^3} \, dx = \frac{1}{3} \int e^u \, du = \frac{1}{3} e^u + C = \frac{1}{3} e^{x^3} + C.
\]

48. \( \int_0^{\ln 3} e^{x-e^x} \, dx \)

**SOLUTION** Note \( e^{x-e^x} = e^x e^{-e^x} \). Now, let \( u = e^x \). Then \( du = e^x \, dx \), and the new limits of integration are \( u = e^0 = 1 \) and \( u = e^{\ln 3} = 3 \). Thus,

\[
\int_0^{\ln 3} e^{x-e^x} \, dx = \int_0^{\ln 3} e^{u-u} \, du = \int_1^3 e^{-u} \, du = -e^{-u} \bigg|_1^3 = -(e^{-3} - e^{-1}) = e^{-1} - e^{-3}.
\]

49. \( \int e^x 10^x \, dx \)

**SOLUTION**

\[
\int e^x 10^x \, dx = \int \frac{(10e)^x}{\ln(10e)} \, dx = \frac{(10e)^x}{\ln 10 + \ln e} + C = \frac{10^x e^x}{\ln 10 + 1} + C.
\]

50. \( \int e^{-2x} \sin(e^{-2x}) \, dx \)

**SOLUTION** Let \( u = e^{-2x} \). Then \( du = -2e^{-2x} \, dx \), and

\[
\int e^{-2x} \sin(e^{-2x}) \, dx = -\frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C = \frac{1}{2} \cos (e^{-2x}) + C.
\]
51. \[ \int \frac{e^{-x} \, dx}{(e^{-x} + 2)^3} \]

**SOLUTION** Let \( u = e^{-x} + 2 \). Then \( du = -e^{-x} \, dx \) and

\[ \int \frac{e^{-x} \, dx}{(e^{-x} + 2)^3} = - \int u^{-3} \, du = \frac{1}{2u^2} + C = \frac{1}{2(e^{-x} + 2)^2} + C. \]

52. \[ \int \sin \theta \cos \theta e^{\cos^2 \theta + 1} \, d\theta \]

**SOLUTION** Let \( u = \cos^2 \theta + 1 \). Then \( du = -2 \sin \theta \cos \theta \, d\theta \) and

\[ \int \sin \theta \cos \theta e^{\cos^2 \theta + 1} \, d\theta = -\frac{1}{2} \int e^u \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{\cos^2 \theta + 1} + C. \]

53. \[ \int_0^{\pi/6} \tan 2\theta \, d\theta \]

**SOLUTION** \[ \int_0^{\pi/6} \tan 2\theta \, d\theta = \frac{1}{2} \ln |\sec 2\theta| \bigg|_0^{\pi/6} = \frac{1}{2} \ln 2. \]

54. \[ \int_{\pi/3}^{2\pi/3} \cot \left( \frac{1}{2} \theta \right) \, d\theta \]

**SOLUTION**

\[ \int_{\pi/3}^{2\pi/3} \cot \left( \frac{1}{2} \theta \right) \, d\theta = 2 \ln \left| \sin \frac{\theta}{2} \right| \bigg|_{\pi/3}^{2\pi/3} = 2 \left( \ln \frac{\sqrt{3}}{2} - \ln \frac{1}{2} \right) = \ln 3. \]

55. \[ \int \frac{dt}{t(1 + (\ln t)^2)} \]

**SOLUTION** Let \( u = \ln t \). Then, \( du = \frac{1}{t} \, dt \) and

\[ \int \frac{dt}{t(1 + (\ln t)^2)} = \int \frac{du}{1 + u^2} = \tan^{-1} u + C = \tan^{-1} (\ln t) + C. \]

56. \[ \int \frac{\cos(\ln x) \, dx}{x} \]

**SOLUTION** Let \( u = \ln x \). Then \( du = \frac{dx}{x} \), and

\[ \int \frac{\cos(\ln x) \, dx}{x} = \int \cos u \, du = \sin u + C = \sin(\ln x) + C. \]

57. \[ \int_1^e \frac{\ln x \, dx}{x} \]

**SOLUTION** Let \( u = \ln x \). Then \( du = \frac{dx}{x} \) and the new limits of integration are \( u = \ln 1 = 0 \) and \( u = \ln e = 1 \). Thus,

\[ \int_1^e \frac{\ln x \, dx}{x} = \int_0^1 u \, du = \frac{1}{2} u^2 \bigg|_0^1 = \frac{1}{2}. \]

58. \[ \int \frac{dx}{x \sqrt{\ln x}} \]

**SOLUTION** Let \( u = \ln x \). Then \( du = \frac{1}{x} \, dx \), and

\[ \int \frac{dx}{x \sqrt{\ln x}} = \int u^{-1/2} \, du = 2 \sqrt{u} + C = 2 \sqrt{\ln x} + C. \]

59. \[ \int \frac{dx}{4x^2 + 9} \]

**SOLUTION** Let \( u = \frac{x}{3} \). Then \( du = \frac{1}{3} \, dx \), and

\[ \int \frac{dx}{4x^2 + 9} = \int \frac{1}{4u^2 + 9} \, du = \frac{1}{2} \tan^{-1} \left( \frac{2u}{3} \right) + C. \]
SOLUTION Let \( u = \frac{2x}{3} \). Then \( x = \frac{3}{2}u, \, dx = \frac{3}{2} \, du \), and
\[
\int \frac{dx}{4x^2 + 9} = \int \frac{\frac{3}{2} \, du}{4 \cdot \frac{3}{2} u^2 + 9} = \frac{1}{6} \int \frac{du}{u^2 + 1} = \frac{1}{6} \tan^{-1} u + C = \frac{1}{6} \tan^{-1} \left( \frac{2x}{3} \right) + C.
\]

60. \( \int_0^0.8 \frac{dx}{\sqrt{1 - x^2}} \)

SOLUTION \( \int_0^0.8 \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \bigg|_0^{0.8} = \sin^{-1} 0.8 - \sin^{-1} 0 = \sin^{-1} 0.8. \)

61. \( \int_4^{12} \frac{dx}{x \sqrt{x^2 - 1}} \)

SOLUTION \( \int_4^{12} \frac{dx}{x \sqrt{x^2 - 1}} = \sec^{-1} x \bigg|_4^{12} = \sec^{-1} 12 - \sec^{-1} 4. \)

62. \( \int_0^3 \frac{x \, dx}{x^2 + 9} \)

SOLUTION Let \( u = x^2 + 9 \). Then \( du = 2x \, dx \), and the new limits of integration are \( u = 9 \) and \( u = 18 \). Thus,
\[
\int_0^3 \frac{x \, dx}{x^2 + 9} = \frac{1}{2} \int_9^{18} \frac{du}{u} = \frac{1}{2} \ln u \bigg|_9^{18} = \frac{1}{2} \ln 18 - \ln 9 = \frac{1}{2} \ln \frac{18}{9} = \frac{1}{2} \ln 2.
\]

63. \( \int_0^3 \frac{dx}{x^2 + 9} \)

SOLUTION Let \( u = \frac{x}{3} \). Then \( du = \frac{dx}{3} \), and the new limits of integration are \( u = 0 \) and \( u = 1 \). Thus,
\[
\int_0^3 \frac{dx}{x^2 + 9} = \frac{1}{3} \int_0^1 \frac{dt}{t^2 + 1} = \frac{1}{3} \tan^{-1} t \bigg|_0^1 = \frac{1}{3} (\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{3} \left( \frac{\pi}{4} - 0 \right) = \frac{\pi}{12}.
\]

64. \( \int \frac{dx}{\sqrt{e^{2x} - 1}} \)

SOLUTION Let \( u = e^x \). Then
\[
du = e^x \, dx = du = u \, dx \Rightarrow u^{-1} du = dx.
\]
By substitution, we obtain
\[
\int \frac{dx}{\sqrt{e^{2x} - 1}} = \int \frac{du}{u \sqrt{u^2 - 1}} = \sec^{-1} u + C = \sec^{-1} (e^x) + C
\]

65. \( \int \frac{x \, dx}{\sqrt{1 - x^2}} \)

SOLUTION Let \( u = x^2 \). Then \( du = 2x \, dx \), and \( \sqrt{1 - x^4} = \sqrt{1 - u^2} \). Thus,
\[
\int \frac{x \, dx}{\sqrt{1 - x^4}} = \frac{1}{2} \int \frac{du}{\sqrt{1 - u^2}} = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} (x^2) + C.
\]

66. \( \int_0^1 \frac{dx}{25 - x^2} \)

SOLUTION Let \( x = 5u \). Then \( dx = 5 \, du \), and the new limits of integration are \( u = 0 \) and \( u = \frac{1}{5} \). Thus,
\[
\int_0^1 \frac{dx}{25 - x^2} = \frac{1}{25} \int_0^{1/5} \frac{5 \, du}{1 - u^2} = \frac{5}{25} \int_0^{1/5} \frac{du}{1 - u^2} = \frac{1}{5} \tanh^{-1} u \bigg|_0^{1/5} = \frac{1}{5} (\tanh^{-1} \frac{1}{5} - \tanh^{-1} 0) = \frac{1}{5} \tanh^{-1} \frac{1}{5}.
\]

67. \( \int_0^4 \frac{dx}{2x^2 + 1} \)
SOLUTION  Let \( u = \sqrt{2}x \). Then \( du = \sqrt{2} \, dx \), and the new limits of integration are \( u = 0 \) and \( u = 4\sqrt{2} \). Thus,

\[
\int_0^4 \frac{dx}{2x^2 + 1} = \int_0^{4\sqrt{2}} \frac{du}{u^2 + 1} = \frac{1}{\sqrt{2}} \int_0^{4\sqrt{2}} \frac{du}{u^2 + 1} = \frac{1}{\sqrt{2}} \tan^{-1} u \bigg|_0^{4\sqrt{2}} = \frac{1}{\sqrt{2}} \left( \tan^{-1}(4\sqrt{2}) - \tan^{-1}0 \right) = \frac{1}{\sqrt{2}} \tan^{-1}(4\sqrt{2}).
\]

68. \( \int_5^8 \frac{dx}{x\sqrt{x^2 - 16}} \)

SOLUTION  Let \( x = 4u \). Then \( dx = 4 \, du \), and the new limits of integration are \( u = \frac{5}{4} \) and \( u = 2 \). Thus,

\[
\int_5^8 \frac{dx}{x\sqrt{x^2 - 16}} = \frac{1}{4} \int_{5/4}^2 \frac{du}{u\sqrt{u^2 - 1}} = \frac{1}{4} \left( \sec^{-1} u \right)^2 \bigg|_{5/4}^2 = \frac{1}{4} \left( \sec^{-1} 2 - \sec^{-1} \frac{5}{4} \right) = \frac{1}{4} \left( \frac{\pi}{3} - \sec^{-1} \frac{5}{4} \right).
\]

69. \( \int_0^1 \frac{(\tan^{-1} x)^3}{1 + x^2} \)

SOLUTION  Let \( u = \tan^{-1} x \). Then

\[
du = \frac{1}{1 + x^2} \, dx
\]

and

\[
\int_0^1 \frac{(\tan^{-1} x)^3}{1 + x^2} \, dx = \int_{\pi/4}^{\pi/4} u^3 \, du = \frac{1}{4} u^4 \bigg|_{\pi/4}^{\pi/4} = \frac{1}{4} \pi^4 = \frac{\pi^4}{1024}.
\]

70. \( \int \frac{\cos^{-1} t \, dt}{\sqrt{1 - t^2}} \)

SOLUTION  Let \( u = \cos^{-1} t \), then \( du = -\frac{1}{\sqrt{1 - t^2}} \, dt \), and

\[
\int \frac{\cos^{-1} t \, dt}{\sqrt{1 - t^2}} = -\int u \, du = -\frac{1}{2} u^2 + C = -\frac{1}{2} (\cos^{-1} t)^2 + C.
\]

71. Combine to write as a single integral:

\[
\int_0^8 f(x) \, dx + \int_{-2}^0 f(x) \, dx + \int_8^6 f(x) \, dx
\]

SOLUTION  First, rewrite

\[
\int_0^8 f(x) \, dx = \int_0^6 f(x) \, dx + \int_8^6 f(x) \, dx
\]

and observe that

\[
\int_8^6 f(x) \, dx = -\int_6^8 f(x) \, dx.
\]

Thus,

\[
\int_0^8 f(x) \, dx + \int_{-2}^0 f(x) \, dx + \int_8^6 f(x) \, dx = \int_0^6 f(x) \, dx.
\]

Finally,

\[
\int_0^8 f(x) \, dx + \int_{-2}^0 f(x) \, dx + \int_8^6 f(x) \, dx = \int_0^6 f(x) \, dx + \int_{-2}^0 f(x) \, dx + \int_8^6 f(x) \, dx.
\]

72. Let \( A(x) = \int_0^x f(x) \, dx \), where \( f(x) \) is the function shown in Figure 4. Identify the location of the local minima, the local maxima, and points of inflection of \( A(x) \) on the interval \([0, E] \), as well as the intervals where \( A(x) \) is increasing, decreasing, concave up, or concave down. Where does the absolute max of \( A(x) \) occur?
**SOLUTION** Let $f(x)$ be the function shown in Figure 4 and define

$$A(x) = \int_0^x f(x) \, dx.$$ 

Then $A'(x) = f(x)$ and $A''(x) = f''(x)$. Hence, $A(x)$ is increasing when $f(x)$ is positive, is decreasing when $f(x)$ is negative, is concave up when $f(x)$ is increasing and is concave down when $f(x)$ is decreasing. Thus, $A(x)$ is increasing for $0 < x < B$, is decreasing for $B < x < D$ and for $D < x < E$, has a local maximum at $x = B$ and no local minima. Moreover, $A(x)$ is concave up for $0 < x < A$ and for $C < x < D$. is concave down for $A < x < C$ and for $D < x < E$, and has a point of inflection at $x = A$, $x = C$ and $x = D$. The absolute maximum value for $A(x)$ occurs at $x = B$.

**73.** Find the local minima, the local maxima, and the inflection points of $A(x) = \int_0^x \frac{t \, dt}{t^2 + 1}$.

**SOLUTION** Let

$$A(x) = \int_0^x \frac{t \, dt}{t^2 + 1}.$$ 

Then

$$A'(x) = \frac{x}{x^2 + 1}$$

and

$$A''(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$ 

Now, $x = 0$ is the only critical point of $A$; because $A''(0) > 0$, it follows that $A$ has a local minimum at $x = 0$. There are no local maxima. Moreover, $A(x)$ is concave down for $|x| > 1$ and concave up for $|x| < 1$. $A(x)$ therefore has inflection points at $x = \pm 1$.

**74.** A particle starts at the origin at time $t = 0$ and moves with velocity $v(t)$ as shown in Figure 5.

(a) How many times does the particle return to the origin in the first 12 seconds?

(b) What is the particle’s maximum distance from the origin?

(c) What is particle’s maximum distance to the left of the origin?

**SOLUTION** Because the particle starts at the origin, the position of the particle is given by

$$s(t) = \int_0^t v(\tau) \, d\tau;$$

that is by the signed area between the graph of the velocity and the $t$-axis over the interval $[0, t]$. Using the geometry in Figure 5, we see that $s(t)$ is increasing for $0 < t < 4$ and for $8 < t < 10$ and is decreasing for $4 < t < 8$ and for $10 < t < 12$. Furthermore,

- $s(0) = 0$ m,
- $s(4) = 4$ m,
- $s(8) = -4$ m,
- $s(10) = -2$ m, and
- $s(12) = -6$ m.

(a) In the first 12 seconds, the particle returns to the origin once, sometime between $t = 4$ and $t = 8$ seconds.

(b) The particle’s maximum distance from the origin is 6 meters (to the left at $t = 12$ seconds).

(c) The particle’s distance to the left of the origin is 6 meters.
75. On a typical day, a city consumes water at the rate of \( r(t) = 100 + 72t - 3t^2 \) (in thousands of gallons per hour), where \( t \) is the number of hours past midnight. What is the daily water consumption? How much water is consumed between 6 PM and midnight?

**SOLUTION** With a consumption rate of \( r(t) = 100 + 72t - 3t^2 \) thousand gallons per hour, the daily consumption of water is

\[
\int_{0}^{24} (100 + 72t - 3t^2) \, dt = \left(100t + 36t^2 - t^3\right)_{0}^{24} = 100(24) + 36(24^2) - (24)^3 = 9312,
\]

or 9.312 million gallons. From 6 PM to midnight, the water consumption is

\[
\int_{18}^{24} (100 + 72t - 3t^2) \, dt = \left(100t + 36t^2 - t^3\right)_{18}^{24} = 100(24) + 36(24^2) - (24)^3 - (100(18) + 36(18^2) - (18)^3)
\]

or 1.68 million gallons.

76. The learning curve in a certain bicycle factory is \( L(x) = 12x^{-1/5} \) (in hours per bicycle), which means that it takes a bike mechanic \( L(n) \) hours to assemble the \( n \)th bicycle. If a mechanic has produced 24 bicycles, how long does it take her or him to produce the second batch of 12?

**SOLUTION** The second batch of 12 bicycles consists of bicycles 13 through 24. The time it takes to produce these bicycles is

\[
\int_{13}^{24} 12x^{-1/5} \, dx = 15x^{4/5}\bigg|_{13}^{24} = 15(24^{4/5} - 13^{4/5}) \approx 73.91 \text{ hours}.
\]

77. Cost engineers at NASA have the task of projecting the cost \( P \) of major space projects. It has been found that the cost \( C \) of developing a projection increases with \( P \) at the rate \( dC/dP = 21P^{-0.65} \), where \( C \) is in thousands of dollars and \( P \) in millions of dollars. What is the cost of developing a projection for a project whose cost turns out to be \( P = 35 \) million?

**SOLUTION** Assuming it costs nothing to develop a projection for a project with a cost of \( 0 \), the cost of developing a projection for a project whose cost turns out to be \( 35 \) million is

\[
\int_{0}^{35} 21P^{-0.65} \, dP = 60P^{0.35}\bigg|_{0}^{35} = 60(35)^{0.35} \approx 208.245,
\]

or $208,245.

78. An astronomer estimates that in a certain constellation, the number of stars per magnitude \( m \), per degree-squared of sky, is equal to \( A(m) = 2.4 \times 10^{-6}m^{-7.4} \) (fainter stars have higher magnitudes). Determine the total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky.

**SOLUTION** The total number of stars of magnitude between 6 and 15 in a one-degree-squared region of sky is

\[
\int_{6}^{15} A(m) \, dm = \int_{6}^{15} 2.4 \times 10^{-6}m^{-7.4} \, dm = \frac{2}{7} \times 10^{-6}m^{8.4}\bigg|_{6}^{15} \approx 2162.
\]

79. Evaluate \( \int_{-8}^{8} \frac{x^{15}}{3 + \cos^2 x} \, dx \), using the properties of odd functions.

**SOLUTION** Let \( f(x) = \frac{x^{15}}{3 + \cos^2 x} \) and note that

\[
f(-x) = \frac{(-x)^{15}}{3 + \cos^2(-x)} = -\frac{x^{15}}{\cos^2 x} = -f(x).
\]

Because \( f(x) \) is an odd function and the interval \(-8 \leq x \leq 8\) is symmetric about \( x = 0 \), it follows that

\[
\int_{-8}^{8} \frac{x^{15}}{3 + \cos^2 x} \, dx = 0.
\]

80. Evaluate \( \int_{0}^{1} f(x) \, dx \), assuming that \( f(x) \) is an even continuous function such that

\[
\int_{1}^{2} f(x) \, dx = 5, \quad \int_{-2}^{1} f(x) \, dx = 8
\]
SOLUTION Using the given information
\[ \int_{-2}^{2} f(x) \, dx = \int_{-2}^{1} f(x) \, dx + \int_{1}^{2} f(x) \, dx = 13. \]
Because \( f(x) \) is an even function, it follows that
\[ \int_{-2}^{0} f(x) \, dx = \int_{0}^{2} f(x) \, dx, \]
so
\[ \int_{0}^{2} f(x) \, dx = \frac{13}{2}. \]
Finally,
\[ \int_{0}^{1} f(x) \, dx = \int_{0}^{2} f(x) \, dx - \int_{1}^{2} f(x) \, dx = \frac{13}{2} - 5 = \frac{3}{2}. \]

81. Plot the graph of \( f(x) = \sin mx \sin nx \) on \([0, \pi]\) for the pairs \((m, n) = (2, 4), (3, 5)\) and in each case guess the value of \( I = \int_{0}^{\pi} f(x) \, dx \). Experiment with a few more values (including two cases with \( m = n \)) and formulate a conjecture for when \( I \) is zero.

SOLUTION The graphs of \( f(x) = \sin mx \sin nx \) with \((m, n) = (2, 4)\) and \((m, n) = (3, 5)\) are shown below. It appears as if the positive areas balance the negative areas, so we expect that
\[ I = \int_{0}^{\pi} f(x) \, dx = 0 \]
in these cases.

We arrive at the same conclusion for the cases \((m, n) = (4, 1)\) and \((m, n) = (5, 2)\).

However, when \((m, n) = (3, 3)\) and when \((m, n) = (5, 5)\), the value of
\[ I = \int_{0}^{\pi} f(x) \, dx \]
is clearly not zero as there is no negative area.

We therefore conjecture that \( I \) is zero whenever \( m \neq n \).
82. Show that
\[
\int x f(x) \, dx = xF(x) - G(x)
\]
where \( F'(x) = f(x) \) and \( G'(x) = F(x) \). Use this to evaluate \( \int x \cos x \, dx \).

**SOLUTION** Suppose \( F'(x) = f(x) \) and \( G'(x) = F(x) \). Then
\[
\frac{d}{dx}(xF(x) - G(x)) = xF'(x) + F(x) - G'(x) = xf(x) + F(x) - F(x) = xf(x).
\]
Therefore, \( xF(x) - G(x) \) is an antiderivative of \( xf(x) \) and
\[
\int xf(x) \, dx = xF(x) - G(x) + C.
\]
To evaluate \( \int x \cos x \, dx \), note that \( f(x) = \cos x \) Thus, we may take \( F(x) = \sin x \) and \( G(x) = -\cos x \). Finally,
\[
\int x \cos x \, dx = x \sin x + \cos x + C.
\]

83. Prove
\[
2 \leq \int_1^2 2^x \, dx \leq 4 \quad \text{and} \quad \frac{1}{9} \leq \int_1^2 3^{-x} \, dx \leq \frac{1}{3}
\]

**SOLUTION** The function \( f(x) = 2^x \) is increasing, so \( 1 \leq x \leq 2 \) implies that \( 2 = 2^1 \leq 2^x \leq 2^2 = 4 \). Consequently,
\[
2 = \int_1^2 2 \, dx \leq \int_1^2 2^x \, dx \leq \int_1^2 4 \, dx = 4.
\]
On the other hand, the function \( f(x) = 3^{-x} \) is decreasing, so \( 1 \leq x \leq 2 \) implies that
\[
\frac{1}{9} = 3^{-2} \leq 3^{-x} \leq 3^{-1} = \frac{1}{3}.
\]
It then follows that
\[
\frac{1}{9} = \int_1^2 \frac{1}{9} \, dx \leq \int_1^2 3^{-x} \, dx \leq \int_1^2 \frac{1}{3} \, dx = \frac{1}{3}.
\]

84. [GU] Plot the graph of \( f(x) = x^{-2} \sin x \), and show that \( 0.2 \leq \int_1^2 f(x) \, dx \leq 0.9 \).

**SOLUTION** Let \( f(x) = x^{-2} \sin x \). From the figure below, we see that
\[
0.2 \leq f(x) \leq 0.9
\]
for \( 1 \leq x \leq 2 \). Therefore,
\[
0.2 \leq \int_1^2 0.2 \, dx \leq \int_1^2 f(x) \, dx \leq \int_1^2 0.9 \, dx = 0.9.
\]

85. Find upper and lower bounds for \( \int_0^1 f(x) \, dx \), for \( f(x) \) in Figure 6.
CHAPTER 5  ■  THE INTEGRAL

### FIGURE 6

![Figure 6](image)

**SOLUTION** From the figure, we see that the inequalities $x^2 + 1 \leq f(x) \leq \sqrt{x} + 1$ hold for $0 \leq x \leq 1$. Because

$$
\int_0^1 (x^2 + 1) \, dx = \left( \frac{1}{3} x^3 + x \right) \bigg|_0^1 = \frac{4}{3}
$$

and

$$
\int_0^1 (\sqrt{x} + 1) \, dx = \left( \frac{2}{3} x^{3/2} + x \right) \bigg|_0^1 = \frac{5}{3},
$$

it follows that

$$
\frac{4}{3} \leq \int_0^1 f(x) \, dx \leq \frac{5}{3}.
$$

*In Exercises 86–91, find the derivative.*

**86.** $A'(x)$, where $A(x) = \int_3^x \sin(t^2) \, dt$

**SOLUTION** Let $A(x) = \int_3^x \sin(t^2) \, dt$. Then $A'(x) = \sin(x^2)$.

**87.** $A'(\pi)$, where $A(x) = \int_2^x \frac{\cos t}{1 + t} \, dt$

**SOLUTION** Let $A(x) = \int_2^x \frac{\cos t}{1 + t} \, dt$. Then $A'(x) = \frac{\cos x}{1 + x}$ and

$$
A'(\pi) = \frac{\cos \pi}{1 + \pi} = -\frac{1}{1 + \pi}.
$$

**88.** $\frac{d}{dy} \int_{-2}^{y} 3^x \, dx$

**SOLUTION**

$$
\frac{d}{dy} \int_{-2}^{y} 3^x \, dx = 3^y.
$$

**89.** $G'(x)$, where $G(x) = \int_{-2}^{\sin x} t^3 \, dt$

**SOLUTION** Let $G(x) = \int_{-2}^{\sin x} t^3 \, dt$. Then

$$
G'(x) = \sin^3 x \frac{d}{dx} \sin x = \sin^3 x \cos x.
$$

**90.** $G'(2)$, where $G(x) = \int_0^{x^3} \sqrt{t + 1} \, dt$

**SOLUTION** Let $G(x) = \int_0^{x^3} \sqrt{t + 1} \, dt$. Then

$$
G'(x) = \sqrt{x^3 + 1} \frac{d}{dx} x^3 = 3x^2 \sqrt{x^3 + 1}
$$

and $G'(2) = (2)^2 \sqrt{8 + 1} = 36$.  

In Exercises 86–91, find the derivative.
91. \( H'(1) \), where \( H(x) = \int_{4x^2}^{9} \frac{1}{t} \, dt \)

**SOLUTION** Let \( H(x) = \int_{4x^2}^{9} \frac{1}{t} \, dt = -\int_{9}^{4x^2} \frac{1}{t} \, dt. \) Then

\[
H'(x) = -\frac{1}{4x^2} \frac{d}{dx} 4x^2 = -\frac{8x}{4x^2} = -\frac{2}{x}
\]

and \( H'(1) = -2. \)

92. Explain with a graph: If \( f(x) \) is increasing and concave up on \([a, b]\), then \( L_N \) is more accurate than \( R_N \). Which is more accurate if \( f(x) \) is increasing and concave down?

**SOLUTION** Consider the figure below, which displays a portion of the graph of an increasing, concave up function.

The shaded rectangles represent the differences between the right-endpoint approximation \( R_N \) and the left-endpoint approximation \( L_N \). In particular, the portion of each rectangle that lies below the graph of \( y = f(x) \) is the amount by which \( L_N \) underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of \( y = f(x) \) is the amount by which \( R_N \) overestimates the area. Because the graph of \( y = f(x) \) is increasing and concave up, the lower portion of each shaded rectangle is smaller than the upper portion. Therefore, \( L_N \) is more accurate (introduces less error) than \( R_N \). By similar reasoning, if \( f(x) \) is increasing and concave down, then \( R_N \) is more accurate than \( L_N \).

93. Explain with a graph: If \( f(x) \) is linear on \([a, b]\), then the \( \int_{a}^{b} f(x) \, dx = \frac{1}{2} (R_N + L_N) \) for all \( N \).

**SOLUTION** Consider the figure below, which displays a portion of the graph of a linear function.

The shaded rectangles represent the differences between the right-endpoint approximation \( R_N \) and the left-endpoint approximation \( L_N \). In particular, the portion of each rectangle that lies below the graph of \( y = f(x) \) is the amount by which \( L_N \) underestimates the area under the graph, whereas the portion of each rectangle that lies above the graph of \( y = f(x) \) is the amount by which \( R_N \) overestimates the area. Because the graph of \( y = f(x) \) is a line, the lower portion of each shaded rectangle is exactly the same size as the upper portion. Therefore, if we average \( L_N \) and \( R_N \), the error in the two approximations will exactly cancel, leaving

\[
\frac{1}{2} (R_N + L_N) = \int_{a}^{b} f(x) \, dx.
\]

94. In this exercise, we prove

\[
x - \frac{x^2}{2} \leq \ln(1 + x) \leq x \quad \text{for } x > 0
\]

(a) Show that \( \ln(1 + x) = \int_{0}^{x} \frac{dt}{1 + t} \) for \( x > 0 \).

(b) Verify that \( 1 - t \leq 1 + t \leq 1 \) for all \( t > 0 \).

(c) Use (b) to prove Eq. (1).

(d) Verify Eq. (1) for \( x = 0.5, 0.1, \) and 0.01.

**SOLUTION**

(a) Let \( x > 0 \). Then

\[
\int_{0}^{x} \frac{dt}{1 + t} = \ln(1 + t) \bigg|_{0}^{x} = \ln(1 + x) - \ln 1 = \ln(1 + x).
\]
(b) For $t > 0$, $1 + t > 0$, so $\frac{1}{1+t} < 1$. Moreover, $(1-t)(1+t) = 1 - t^2 < 1$. Because $1 + t > 0$, it follows that $1 - t < \frac{1}{1+t}$. Hence,

$$1 - t \leq \frac{1}{1+t} \leq 1.$$ 

(c) Integrating each expression in the result from part (b) from $t = 0$ to $t = x$ yields

$$x - \frac{x^2}{2} \leq \ln(1 + x) \leq x.$$ 

(d) For $x = 0.5$, $x = 0.1$ and $x = 0.01$, we obtain the string of inequalities

$$0.375 \leq 0.405465 \leq 0.5$$
$$0.095 \leq 0.095310 \leq 0.1$$
$$0.00995 \leq 0.00995033 \leq 0.01,$$

respectively.

95. Let

$$F(x) = x \sqrt{x^2 - 1} - 2 \int_1^x \sqrt{t^2 - 1} \, dt$$

Prove that $F(x)$ and $\cosh^{-1} x$ differ by a constant by showing that they have the same derivative. Then prove they are equal by evaluating both at $x = 1$.

**SOLUTION** Let

$$F(x) = x \sqrt{x^2 - 1} - 2 \int_1^x \sqrt{t^2 - 1} \, dt.$$ 

Then

$$\frac{dF}{dx} = \sqrt{x^2 - 1} + \frac{x^2}{\sqrt{x^2 - 1}} - 2\sqrt{x^2 - 1} = \frac{x^2}{\sqrt{x^2 - 1}} - \sqrt{x^2 - 1} = \frac{1}{\sqrt{x^2 - 1}}.$$ 

Also, $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$; therefore, $F(x)$ and $\cosh^{-1} x$ have the same derivative. We conclude that $F(x)$ and $\cosh^{-1} x$ differ by a constant:

$$F(x) = \cosh^{-1} x + C.$$ 

Now, let $x = 1$. Because $F(1) = 0$ and $\cosh^{-1} 1 = 0$, it follows that $C = 0$. Therefore,

$$F(x) = \cosh^{-1} x.$$ 

96. Let $f(x)$ be a positive increasing continuous function on $[a, b]$, where $0 \leq a < b$ as in Figure 7. Show that the shaded region has area

$$I = bf(b) - af(a) - \int_a^b f(x) \, dx$$

**FIGURE 7**

**SOLUTION** We can construct the shaded region in Figure 7 by taking a rectangle of length $b$ and height $f(b)$ and removing a rectangle of length $a$ and height $f(a)$ as well as the region between the graph of $y = f(x)$ and the $x$-axis over the interval $[a, b]$. The area of the resulting region is then the area of the large rectangle minus the area of the small rectangle and minus the area under the curve $y = f(x)$; that is,

$$I = bf(b) - af(a) - \int_a^b f(x) \, dx.$$
97. How can we interpret the quantity $I$ in Eq. (2) if $a < b \leq 0$? Explain with a graph.

**SOLUTION** We will consider each term on the right-hand side of (2) separately. For convenience, let $I, II, III$ and $IV$ denote the area of the similarly labeled region in the diagram below.

Because $b < 0$, the expression $bf(b)$ is the opposite of the area of the rectangle along the right; that is,

$$bf(b) = -II - IV.$$ 

Similarly,

$$-af(a) = III + IV \quad \text{and} \quad \int_a^b f(x) \, dx = -I - III.$$ 

Therefore,

$$bf(b) - af(a) - \int_a^b f(x) \, dx = -I - II;$$

that is, the opposite of the area of the shaded region shown below.

98. The isotope thorium-234 has a half-life of 24.5 days.

(a) What is the differential equation satisfied by $y(t)$, the amount of thorium-234 in a sample at time $t$?

(b) At $t = 0$, a sample contains 2 kg of thorium-234. How much remains after 40 days?

**SOLUTION**

(a) By the equation for half-life,

$$24.5 = \frac{\ln 2}{k}, \quad \text{so} \quad k = \frac{\ln 2}{24.5} \approx 0.028 \text{ days}^{-1}.$$ 

Therefore, the differential equation for $y(t)$ is

$$y' = -0.028y.$$ 

(b) If there are 2 kg of thorium-234 at $t = 0$, then $y(t) = 2e^{-0.028t}$. After 40 days, the amount of thorium-234 is

$$y(40) = 2e^{-0.028(40)} = 0.653 \text{ kg}.$$ 

99. The Oldest Snack Food? In Bat Cave, New Mexico, archaeologists found ancient human remains, including cobs of popping corn whose $^{14}\text{C}$-to-$^{12}\text{C}$ ratio was approximately 48% of that found in living matter. Estimate the age of the corn cobs.

**SOLUTION** Let $t$ be the age of the corn cobs. The $^{14}\text{C}$ to $^{12}\text{C}$ ratio decreased by a factor of $e^{-0.000121t}$ which is equal to 0.48. That is:

$$e^{-0.000121t} = 0.48,$$

so

$$-0.000121t = \ln 0.48,$$

and

$$t = \frac{1}{0.000121} \ln 0.48 \approx 6065.9.$$ 

We conclude that the age of the corn cobs is approximately 6065.9 years.
100. The $^{14}$C-$^{12}$ ratio of a sample is proportional to the disintegration rate (number of beta particles emitted per minute) that is measured directly with a Geiger counter. The disintegration rate of carbon in a living organism is 15.3 beta particles per minute per gram. Find the age of a sample that emits 9.5 beta particles per minute per gram.

**SOLUTION** Let $t$ be the age of the sample in years. Because the disintegration rate for the sample has dropped from 15.3 beta particles/min per gram to 9.5 beta particles/min per gram and the $^{14}$C to $^{12}$C ratio is proportional to the disintegration rate, it follows that

$$e^{-0.000121t} = \frac{9.5}{15.3},$$

so

$$t = -\frac{1}{0.000121} \ln \frac{9.5}{15.3} \approx 3938.5.$$

We conclude that the sample is approximately 3938.5 years old.

101. What is the interest rate if the PV of $50,000 to be delivered in 3 years is $43,000?

**SOLUTION** Let $r$ denote the interest rate. The present value of $50,000 received in 3 years with an interest rate of $r$ is $50,000e^{-3r}$. Thus, we need to solve

$$43,000 = 50,000e^{-3r}$$

for $r$. This yields

$$r = -\frac{1}{3} \ln \frac{43}{50} = 0.0503.$$

Thus, the interest rate is 5.03%.

102. An equipment upgrade costing $1 million will save a company $320,000 per year for 4 years. Is this a good investment if the interest rate is 5%? What is the largest interest rate that would make the investment worthwhile? Assume that the savings are received as a lump sum at the end of each year.

**SOLUTION** With an interest rate of $r = 5\%$, the present value of the four payments is

$$\$320,000(e^{-0.05} + e^{-0.1} + e^{-0.15} + e^{-0.2}) = \$1,131,361.78.$$

As this is greater than the $1 million cost of the upgrade, this is a good investment. To determine the largest interest rate that would make the investment worthwhile, we must solve the equation

$$320,000(e^{-r} + e^{-2r} + e^{-3r} + e^{-4r}) = 1,000,000$$

for $r$. Using a computer algebra system, we find $r = 10.13\%$.

103. Find the PV of an income stream paying out continuously at a rate of $5000e^{-0.14t}$ dollars per year for 5 years, assuming an interest rate of $r = 4\%$.

**SOLUTION** $PV = \int_{0}^{5} 5000e^{-0.14t}dt = 5000e^{-0.14t} \bigg|_{0}^{5} = \frac{5000}{0.14}e^{-0.14t} \bigg|_{0}^{5} = \$17,979.10.$$

104. Calculate the limit:

(a) $\lim_{n \to \infty} \left( 1 + \frac{4}{n} \right)^n$

(b) $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{4n}$

(c) $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{3n}$

**SOLUTION**

(a) $\lim_{n \to \infty} \left( 1 + \frac{4}{n} \right)^n = \lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n/4} \right)^{n/4} \right]^4 = e^4.$

(b) $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{4n} = \lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n} \right)^n \right]^4 = e^4.$

(c) $\lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{3n} = \lim_{n \to \infty} \left[ \left( 1 + \frac{1}{n/4} \right)^{n/4} \right]^{12} = e^{12}.$