Chapter 5: The Integral
Preparing for the AP Exam Solutions

Multiple Choice Questions

1) B  2) D  3) E  4) D  5) C  
6) B  7) A  8) C  9) C  10) A  

Free Response Questions

1.  a) If $v(t) > 0$, then $x(t)$ will be increasing, so set $\frac{1}{2} - \sin t > 0$. Solution is $0 \leq t < \frac{\pi}{6}$ and $\frac{5\pi}{6} < t \leq 2\pi$  
   
b) $3 + \int_{0}^{2\pi} \left(\frac{1}{2} - \sin t\right) dt = 3 + \pi$  
   
c) $\int_{0}^{\pi/6} \left(\frac{1}{2} - \sin t\right) dt + \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} - \sin t\right) dt + \int_{5\pi/6}^{2\pi} \left(\frac{1}{2} - \sin t\right) dt = 2\sqrt{3} + \frac{\pi}{3}$  
   
d) When $t = \frac{\pi}{4}$, $v(t) = \frac{1}{2} - \frac{\sqrt{2}}{2} < 0$ and $a(t) = -\cos t = -\frac{\sqrt{2}}{2} < 0$. $v(t)$ is negative and decreasing, so $|v(t)|$, or the speed, is increasing.

POINTS:
(a) (3 pts) 1) $\frac{1}{2} - \sin t > 0$; 1) $0 \leq t < \frac{\pi}{6}$; 1) $\frac{5\pi}{6} < t \leq 2\pi$
(b) (1 pt)
(c) (3 pts) 1) integrates $v(t)$ over answer to part (a); 1) integrates $-v(t)$ over complement; 1) answer
(d) (2 pts) 1) $v(\frac{\pi}{4}) < 0$ and $a(\frac{\pi}{4}) < 0$; 1) Answer

2.  a) We need $\sqrt{t^3 + 64}$ defined on the interval whose endpoints are 0 and $x^2$. Since $x^2 > 0$ for all $x$, the domain is the entire number line.
   
b) $g'(x) = \sqrt{(x^2)^3 + 64 \cdot (2x)} = 2x\sqrt{x^6 + 64} > 0$ for $x > 0$. Thus, since $g$ is continuous at 0, $g(x)$ is increasing on $[0, \infty)$.
   
c) $g''(x) = 2\sqrt{x^6 + 64} + 2x \frac{1}{2\sqrt{x^6 + 64}} 6x^5$. Thus $g''(0) = 16$

POINTS:
(a) (2 pts) 1) $x^2 > 0$; 1) Answer
(b) (4 pts) 2) $g'(x) = \sqrt{(x^2)^3 + 64 \cdot (2x)}$ Note: 1 pt for $\sqrt{(x^2)^3 + 64}$, 1 pt for chain rule; 1) Sets $g'(x) > 0$; 1) Answer
(c) (3 pts) 2) $g''(x)$; 1) $g''(0)$
3. a) $g$ has a local maximum when $g'(x) = f(x)$ changes from positive to negative; this happens when $x = 4$.

b) The maximum occurs either at a local maximum, or at an end point. $g(4) = \frac{1}{2} \cdot 2 \cdot 4 = 4$, the area of the triangle; $g$ decreases from 4 to 5, so we only need to check

$$g(-3) = \int_{-3}^{2} f(x)dx = -\int_{-3}^{0} f(x)dx = -(\int_{-3}^{0} f(x)dx + \int_{0}^{2} f(x)dx) = -(-9 + 4) = 5.$$ The maximum value of $g(x)$ is 5.

c) The graph of $g$ is concave up when $g'(x)$ is increasing, that is on $(-3, 2)$.

POINTS:
(a) (3 pts) 1) Identifies $g'(x) = f(x)$; 1) $x = 4$; 1) justification
(b) (4 pts) 1) Evaluates $g(4)$; 1) deals with left end point; 1) deals with right end point; 1) answer
(c) (2 pts) 1) answer; 2) justification

4. a) $g(0) = \int_{0}^{1} f(t)dt = -\int_{0}^{1} f(t)dt = -\frac{1}{4} \pi \cdot (1)^2 = -\frac{\pi}{4}$

b) $g'(x)$ exists for all $x$ because $f$ is continuous.

c) $g''(x)$ fails to exist at $x = 2$ and 6 because $g''(x) = f''(x)$ and $f$ is not differentiable at 2 and 6.

d) $g(0) = -\frac{\pi}{4}$; $g$ increases from 0 to 2.

$g(2) = \frac{\pi}{4}$; $g$ decreases from 2 to 6.

$g(6) = \frac{\pi}{4} - \frac{1}{2} \pi (2)^2 = -\frac{7\pi}{4}$; $g$ increases from 6 to 10.

$g(10) = \frac{7\pi}{4} + (\frac{1}{2})(4)(4) = -\frac{7\pi}{4} + 8 > 0$

$g(x) = 0$ has three solutions, one each in $(0, 2)$, $(2, 6)$, and $(6, 10)$.

POINTS:
(a) (1 pt) Answer
(b) (2 pts) 1) Answer; 1) $f$ is continuous
(c) (3 pts) 1) $g'' = f''$; 1) $x = 2$ and 6; 1) $f$ not differentiable
(d) (3 pts) 1) Finds $g(2)$ and $g(6)$; 1) Finds $g(0)$ and $g(10)$; 2) Uses sign changes of $g$