34. Let \( f(x) = 1 \) if \( x \) is rational and \( f(x) = 0 \) if \( x \) is irrational. Prove that \( \lim_{x \to c} f(x) \) does not exist for any \( c \).

**SOLUTION** Let \( c \) be any number, and let \( \delta > 0 \) be an arbitrary small number. We will prove that there is an \( x \) such that \( |x - c| < \delta \), but \( |f(x) - f(c)| > \frac{1}{\delta} \). \( c \) must be either irrational or rational. If \( c \) is rational, then \( f(c) = 1 \). Since the irrational numbers are dense, there is at least one irrational number \( z \) such that \( |z - c| < \delta \). \( |f(z) - f(c)| = 1 > \frac{1}{\delta} \), so the function is discontinuous at \( x = c \).

On the other hand, if \( c \) is irrational, then there is a rational number \( q \) such that \( |q - c| < \delta \). \( |f(q) - f(c)| = |1 - 0| = 1 > \frac{1}{\delta} \), so the function is discontinuous at \( x = c \).

35. Here is a function with strange continuity properties:

\[
f(x) = \begin{cases} 
\frac{1}{q} & \text{if } x \text{ is the rational number } p/q \text{ in lowest terms} \\
0 & \text{if } x \text{ is an irrational number}
\end{cases}
\]

(a) Show that \( f(x) \) is discontinuous at \( c \) if \( c \) is rational. **Hint:** There exist irrational numbers arbitrarily close to \( c \).

(b) Show that \( f(x) \) is continuous at \( c \) if \( c \) is irrational. **Hint:** Let \( I \) be the interval \( \{x : |x - c| < 1\} \). Show that for any \( Q > 0 \), \( I \) contains at most finitely many fractions \( p/q \) with \( q < Q \). Conclude that there is a \( \delta \) such that all fractions in \( \{x : |x - c| < \delta\} \) have a denominator larger than \( Q \).

**SOLUTION**

(a) Let \( c \) be any rational number and suppose that, in lowest terms, \( c = p/q \), where \( p \) and \( q \) are integers. To prove the discontinuity of \( f \) at \( c \), we must show there is an \( \epsilon > 0 \) such that for any \( \delta > 0 \) there is an \( x \) for which \( |x - c| < \delta \), but that \( |f(x) - f(c)| > \epsilon \). Let \( \epsilon = \frac{1}{2q} \) and \( \delta > 0 \). Since there is at least one irrational number between any two distinct real numbers, there is some irrational \( x \) between \( c \) and \( c + \delta \). Hence, \( |x - c| < \delta \), but \( |f(x) - f(c)| = |0 - \frac{1}{q}| = \frac{1}{q} > \frac{1}{2q} = \epsilon \).

(b) Let \( c \) be irrational, let \( \epsilon > 0 \) be given, and let \( N > 0 \) be a prime integer sufficiently large so that \( \frac{1}{N} < \epsilon \). Let \( \frac{p_1}{q_1}, \ldots, \frac{p_m}{q_m} \) be all rational numbers \( \frac{p}{q} \) in lowest terms such that \( \frac{|p_i}{q_i} - c| < 1 \) and \( q < N \). Since \( N \) is finite, this is a finite list; hence, one number \( \frac{p_i}{q_i} \) in the list must be closest to \( c \). Let \( \delta = \frac{1}{2} \frac{|p_i|}{q_i} - c \). By construction, \( |\frac{p_i}{q_i} - c| > \delta \) for all \( i = 1, \ldots, m \). Therefore, for any rational number \( \frac{p}{q} \) such that \( |\frac{p}{q} - c| < \delta, q > N \), so \( \frac{p}{q} < \frac{p_i}{q_i} < c \).

Therefore, for any rational number \( x \) such that \( |x - c| < \delta \), \( |f(x) - f(c)| < \epsilon \). |f(x) - f(c)| = 0 for any irrational number \( x \), so \( |x - c| < \delta \) implies that \( |f(x) - f(c)| < \epsilon \) for any number \( x \).

**CHAPTER REVIEW EXERCISES**

1. The position of a particle at time \( t \) (s) is \( s(t) = \sqrt{t^2 + 1} \) m. Compute its average velocity over \([2, 5]\) and estimate its instantaneous velocity at \( t = 2 \).

**SOLUTION** Let \( s(t) = \sqrt{t^2 + 1} \). The average velocity over \([2, 5]\) is

\[
\frac{s(5) - s(2)}{5 - 2} = \frac{\sqrt{26} - \sqrt{5}}{3} \approx 0.954 \text{ m/s}.
\]

From the data in the table below, we estimate that the instantaneous velocity at \( t = 2 \) is approximately 0.894 m/s.

<table>
<thead>
<tr>
<th>interval</th>
<th>[1.9, 2]</th>
<th>[1.99, 2]</th>
<th>[1.999, 2]</th>
<th>[2.2, 001]</th>
<th>[2.2, 01]</th>
<th>[2.2, 1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>average ROC</td>
<td>0.889769</td>
<td>0.893978</td>
<td>0.894382</td>
<td>0.894472</td>
<td>0.894873</td>
<td>0.898727</td>
</tr>
</tbody>
</table>

2. The “wellhead” price \( p \) of natural gas in the United States (in dollars per 1000 ft\(^3\)) on the first day of each month in 2008 is listed in the table below.

<table>
<thead>
<tr>
<th></th>
<th>J</th>
<th>F</th>
<th>M</th>
<th>A</th>
<th>M</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>price</td>
<td>6.99</td>
<td>7.55</td>
<td>8.29</td>
<td>8.94</td>
<td>9.81</td>
<td>10.82</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>J</th>
<th>A</th>
<th>S</th>
<th>O</th>
<th>N</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>price</td>
<td>10.62</td>
<td>8.32</td>
<td>7.27</td>
<td>6.36</td>
<td>5.97</td>
<td>5.87</td>
</tr>
</tbody>
</table>

Compute the average rate of change of \( p \) (in dollars per 1000 ft\(^3\) per month) over the quarterly periods January–March, April–June, and July–September.

**SOLUTION** To determine the average rate of change in price over the first quarter, divide the difference between the April and January prices by the three-month duration of the quarter. This yields

\[
\frac{8.94 - 6.99}{3} = 0.65 \text{ dollars per 1000 ft}^3 \text{ per month}.
\]
In a similar manner, we calculate the average rates of change for the second and third quarters of the year to be

\[ \frac{10.62 - 8.94}{3} = 0.56 \text{ dollars per 1000 ft}^3 \text{ per month.} \]

and

\[ \frac{6.36 - 10.62}{3} = -1.42 \text{ dollars per 1000 ft}^3 \text{ per month.} \]

3. For a whole number \( n \), let \( P(n) \) be the number of partitions of \( n \), that is, the number of ways of writing \( n \) as a sum of one or more whole numbers. For example, \( P(4) = 5 \) since the number 4 can be partitioned in five different ways: \( 4, 3 + 1, 2 + 2, 2 + 1 + 1, \) and \( 1 + 1 + 1 + 1 \). Treating \( P(n) \) as a continuous function, use Figure 1 to estimate the rate of change of \( P(n) \) at \( n = 12 \).

\[ P(n) \]

\[ \begin{array}{c|cccccc}
\text{average ROC} & 0.744256 & 0.744199 & 0.744193 & 0.744195 & 0.744187 & 0.744131 \\
\end{array} \]

In Exercises 5–10, estimate the limit numerically to two decimal places or state that the limit does not exist.

5. \( \lim_{x \to 0} \frac{1 - \cos^3(x)}{x^2} \)

**SOLUTION** Let \( f(x) = \frac{1 - \cos^3(x)}{x^2} \). The data in the table below suggests that

\[ \lim_{x \to 0} \frac{1 - \cos^3(x)}{x^2} \approx 1.50. \]

In constructing the table, we take advantage of the fact that \( f \) is an even function.

\[ \begin{array}{c|ccc}
x & \pm 0.001 & \pm 0.01 & \pm 0.1 \\
f(x) & 1.500000 & 1.499912 & 1.491275 \\
\end{array} \]

(The exact value is \( \frac{3}{2} \).)

6. \( \lim_{x \to 1} x^{1/(x-1)} \)

**SOLUTION** Let \( f(x) = x^{1/(x-1)} \). The data in the table below suggests that

\( \lim_{x \to 1} x^{1/(x-1)} \approx 2.72. \)

\[ \begin{array}{c|cccccc}
x & 0.9 & 0.99 & 0.999 & 1.001 & 1.01 & 1.1 \\
f(x) & 2.867972 & 2.731999 & 2.719642 & 2.716924 & 2.704814 & 2.593742 \\
\end{array} \]

(The exact value is \( e \).)
7. \( \lim_{x \to 2} \frac{x^4 - 4}{x^2 - 4} \)

**SOLUTION** Let \( f(x) = \frac{x^4 - 4}{x^2 - 4} \). The data in the table below suggests that

\[
\lim_{x \to 2} \frac{x^4 - 4}{x^2 - 4} \approx 1.69.
\]

<table>
<thead>
<tr>
<th>x</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>1.575461</td>
<td>1.680633</td>
<td>1.691888</td>
<td>1.694408</td>
<td>1.705836</td>
<td>1.828386</td>
</tr>
</tbody>
</table>

(The exact value is \( 1 + \ln 2 \).)

8. \( \lim_{x \to 2} \frac{x - 2}{\ln(3x - 5)} \)

**SOLUTION** Let \( f(x) = \frac{x - 2}{\ln(3x - 5)} \). The data in the table below suggests that

\[
\lim_{x \to 2} \frac{x - 2}{\ln(3x - 5)} \approx 0.33.
\]

<table>
<thead>
<tr>
<th>x</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.280367</td>
<td>0.328308</td>
<td>0.332833</td>
<td>0.333833</td>
<td>0.338309</td>
<td>0.381149</td>
</tr>
</tbody>
</table>

(The exact value is \( 1/3 \).)

9. \( \lim_{x \to 1} \left( \frac{7}{1-x^3} - \frac{3}{1-x^3} \right) \)

**SOLUTION** Let \( f(x) = \left( \frac{7}{1-x^3} - \frac{3}{1-x^3} \right) \). The data in the table below suggests that

\[
\lim_{x \to 1} \left( \frac{7}{1-x^3} - \frac{3}{1-x^3} \right) \approx 2.00.
\]

<table>
<thead>
<tr>
<th>x</th>
<th>0.9</th>
<th>0.99</th>
<th>0.999</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2.347483</td>
<td>2.033498</td>
<td>2.003335</td>
<td>1.996668</td>
<td>1.966835</td>
<td>1.685059</td>
</tr>
</tbody>
</table>

(The exact value is 2.)

10. \( \lim_{x \to 2} \frac{3^x - 9}{5^x - 25} \)

**SOLUTION** Let \( f(x) = \frac{3^x - 9}{5^x - 25} \). The data in the table below suggests that

\[
\lim_{x \to 2} \frac{3^x - 9}{5^x - 25} \approx 0.246.
\]

<table>
<thead>
<tr>
<th>x</th>
<th>1.9</th>
<th>1.99</th>
<th>1.999</th>
<th>2.001</th>
<th>2.01</th>
<th>2.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.251950</td>
<td>0.246365</td>
<td>0.245801</td>
<td>0.245675</td>
<td>0.245110</td>
<td>0.239403</td>
</tr>
</tbody>
</table>

(The exact value is \( \frac{3 \ln 3}{25 \ln 5} \).)

*In Exercises 11–50, evaluate the limit if it exists. If not, determine whether the one-sided limits exist (finite or infinite).*

11. \( \lim_{x \to 4} (3 + x^{1/2}) \)

**SOLUTION** \( \lim_{x \to 4} (3 + x^{1/2}) = 3 + \sqrt{7} = 5. \)

12. \( \lim_{x \to 1} \frac{5 - x^2}{4x + 7} \)

**SOLUTION** \( \lim_{x \to 1} \frac{5 - x^2}{4x + 7} = \frac{5 - 1^2}{4(1) + 7} = \frac{4}{11} \).

13. \( \lim_{x \to 2} \frac{4}{x^3} \)
19. \lim_{x \to -2} x^3 = \frac{4}{(-2)^3} = \frac{1}{2}.

14. \lim_{x \to 1} \frac{3x^2 + 4x + 1}{x + 1}

SOLUTION \lim_{x \to -1} \frac{3x^2 + 4x + 1}{x + 1} = \lim_{x \to -1} \frac{(3x + 1)(x + 1)}{x + 1} = \lim_{x \to -1} (3x + 1) = 3(-1) + 1 = -2.

15. \lim_{t \to 9} \frac{\sqrt{t} - 3}{t - 9}

SOLUTION \lim_{t \to 9} \frac{\sqrt{t} - 3}{t - 9} = \lim_{t \to 9} \frac{\sqrt{t} - 3}{\sqrt{t} - 3(\sqrt{t} + 3)} = \lim_{t \to 9} \frac{1}{\sqrt{t} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}.

16. \lim_{x \to 3} \frac{\sqrt{x} + 1 - 2}{x - 3}

SOLUTION

\begin{align*}
\lim_{x \to 3} \frac{\sqrt{x} + 1 - 2}{x - 3} &= \lim_{x \to 3} \frac{\sqrt{x} + 1 - 2}{x - 3} \\
&= \lim_{x \to 3} \frac{\sqrt{x} + 1 + 2}{x - 3} = \lim_{x \to 3} \frac{(x + 1) - 4}{(x - 3)(\sqrt{x} + 1 + 2)} \\
&= \lim_{x \to 3} \frac{1}{\sqrt{x} + 1 + 2} = \frac{1}{3 + 1 + 2} = \frac{1}{4}.
\end{align*}

17. \lim_{x \to 1} \frac{x^3 - x}{x - 1}

SOLUTION \lim_{x \to 1} \frac{x^3 - x}{x - 1} = \lim_{x \to 1} \frac{x(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} x(x + 1) = 1(1 + 1) = 2.

18. \lim_{h \to 0} \frac{2(a + h)^2 - 2a^2}{h}

SOLUTION

\begin{align*}
\lim_{h \to 0} \frac{2(a + h)^2 - 2a^2}{h} &= \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 - 2a^2}{h} = \lim_{h \to 0} \frac{h(4a + 2h)}{h} = \lim_{h \to 0} (4a + 2h) = 4a + 2(0) = 4a.
\end{align*}

19. \lim_{t \to 9} \frac{t - 6}{\sqrt{t} - 3}

SOLUTION Because the one-sided limits

\[
\lim_{t \to 9^+} \frac{t - 6}{\sqrt{t} - 3} = -\infty \quad \text{and} \quad \lim_{t \to 9^-} \frac{t - 6}{\sqrt{t} - 3} = \infty,
\]

are not equal, the two-sided limit

\[
\lim_{t \to 9} \frac{t - 6}{\sqrt{t} - 3}
\]

does not exist.

20. \lim_{s \to 0} \frac{1 - \sqrt{s^2 + 1}}{s^2}

SOLUTION

\begin{align*}
\lim_{s \to 0} \frac{1 - \sqrt{s^2 + 1}}{s^2} &= \lim_{s \to 0} \frac{1 - \sqrt{s^2 + 1}}{s^2} \cdot \frac{1 + \sqrt{s^2 + 1}}{1 + \sqrt{s^2 + 1}} = \lim_{s \to 0} \frac{1 - (s^2 + 1)}{s^2(1 + \sqrt{s^2 + 1})} \\
&= \lim_{s \to 0} \frac{-1}{1 + \sqrt{s^2 + 1}} = \frac{-1}{1 + \sqrt{0^2 + 1}} = \frac{-1}{2}.
\end{align*}

21. \lim_{x \to 1^+} \frac{1}{x + 1}

SOLUTION For \(x > -1, x + 1 > 0\). Therefore,

\[
\lim_{x \to 1^+} \frac{1}{x + 1} = \infty.
\]

22. \lim_{y \to 4} \frac{3y^2 + 5y - 2}{6y^2 - 5y + 1}

SOLUTION
SOLUTION

\[ \lim_{y \to \frac{3}{4}} \frac{3y^2 + 5y - 2}{6y^2 - 5y + 1} = \lim_{y \to \frac{3}{4}} \frac{(3y - 1)(y + 2)}{(3y - 1)(2y - 1)} = \lim_{y \to \frac{3}{4}} \frac{y + 2}{2y - 1} = -7. \]

23. \[ \lim_{x \to 1} \frac{x^3 - 2x}{x - 1} \]

SOLUTION Because the one-sided limits

\[ \lim_{x \to 1^-} \frac{x^3 - 2x}{x - 1} = \infty \quad \text{and} \quad \lim_{x \to 1^+} \frac{x^3 - 2x}{x - 1} = -\infty, \]

are not equal, the two-sided limit

\[ \lim_{x \to 1} \frac{x^3 - 2x}{x - 1} \]

does not exist.

24. \[ \lim_{a \to b} \frac{a^2 - 3ab + 2b^2}{a - b} \]

SOLUTION

\[ \lim_{a \to b} \frac{a^2 - 3ab + 2b^2}{a - b} = \lim_{a \to b} \frac{(a - b)(a - 2b)}{a - b} = \lim_{a \to b} (a - 2b) = b - 2b = -b. \]

25. \[ \lim_{x \to 0} \frac{e^{3x} - e^x}{e^x - 1} \]

SOLUTION

\[ \lim_{x \to 0} \frac{e^{3x} - e^x}{e^x - 1} = \lim_{x \to 0} \frac{e^x(e^x - 1)(e^x + 1)}{e^x - 1} = \lim_{x \to 0} e^x(e^x + 1) = 1 \cdot 2 = 2. \]

26. \[ \lim_{\theta \to 0} \frac{\sin 5\theta}{\theta} \]

SOLUTION

\[ \lim_{\theta \to 0} \frac{\sin 5\theta}{\theta} = 5 \lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta} = 5(1) = 5. \]

27. \[ \lim_{x \to 1.5} [x] \]

SOLUTION

\[ \lim_{x \to 1.5} [x] = [1.5] = 1 \quad \text{at} \quad 1.5 = \frac{2}{3}. \]

28. \[ \lim_{\theta \to \frac{\pi}{4}} \sec \theta \]

SOLUTION

\[ \lim_{\theta \to \frac{\pi}{4}} \sec \theta = \sec \frac{\pi}{4} = \sqrt{2}. \]

29. \[ \lim_{z \to 3} \frac{z + 3}{z^2 + 4z + 3} \]

SOLUTION

\[ \lim_{z \to 3} \frac{z + 3}{z^2 + 4z + 3} = \lim_{z \to 3} \frac{z + 3}{(z + 3)(z + 1)} = \lim_{z \to 3} \frac{1}{z + 1} = \frac{1}{2}. \]

30. \[ \lim_{x \to 1} \frac{x^3 - ax^2 + ax - 1}{x - 1} \]

SOLUTION Using

\[ x^3 - ax^2 + ax - 1 = (x - 1)(x^2 + x + 1) - ax(x - 1) = (x - 1)(x^2 + x - ax + 1) \]

we find

\[ \lim_{x \to 1} \frac{x^3 - ax^2 + ax - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x - ax + 1)}{x - 1} = \lim_{x \to 1} (x^2 + x - ax + 1) \]

\[ = 1^2 + 1 - a(1) + 1 = 3 - a. \]
31. \( \lim_{x \to b} \frac{x^3 - b^3}{x - b} \)

**SOLUTION**

\[
\lim_{x \to b} \frac{x^3 - b^3}{x - b} = \lim_{x \to b} \frac{(x - b)(x^2 + xb + b^2)}{x - b} = \lim_{x \to b} (x^2 + xb + b^2) = b^2 + b(b) + b^2 = 3b^2.
\]

32. \( \lim_{x \to 0} \frac{\sin 4x}{\sin 3x} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} = \frac{4}{3} \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{3x}{\sin 3x} = \frac{4}{3} \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \lim_{x \to 0} \frac{3x}{3x} = \frac{4}{3}(1)(1) = \frac{4}{3}.
\]

33. \( \lim_{x \to 0} \left( \frac{1}{3x} - \frac{1}{x(x + 3)} \right) \)

**SOLUTION**

\[
\lim_{x \to 0} \left( \frac{1}{3x} - \frac{1}{x(x + 3)} \right) = \lim_{x \to 0} \frac{(x + 3) - 3}{3x(x + 3)} = \lim_{x \to 0} \frac{1}{3(x + 3)} = \frac{1}{3} = \frac{1}{3}.
\]

34. \( \lim_{\theta \to \frac{\pi}{4}} \tan(\pi \theta) \)

**SOLUTION**

\[
\lim_{\theta \to \frac{\pi}{4}} \tan(\pi \theta) = \tan(\pi \cdot \frac{\pi}{4}) = \tan(\frac{\pi^2}{4}) = 3^\pi = 3.
\]

35. \( \lim_{x \to 0^+} \frac{[x]}{x} \)

**SOLUTION**

For \( x \) sufficiently close to zero but negative, \([x] = -1\). Therefore,

\[
\lim_{x \to 0^-} \frac{[x]}{x} = \lim_{x \to 0^-} \frac{-1}{x} = \infty.
\]

36. \( \lim_{x \to 0^-} \frac{[x]}{x} \)

**SOLUTION**

For \( x \) sufficiently close to zero but positive, \([x] = 0\). Therefore,

\[
\lim_{x \to 0^+} \frac{[x]}{x} = \lim_{x \to 0^+} \frac{0}{x} = 0.
\]

37. \( \lim_{\theta \to \frac{\pi}{4}} \theta \sec \theta \)

**SOLUTION**

Because the one-sided limits

\[
\lim_{\theta \to \frac{\pi}{4}^-} \theta \sec \theta = \infty \quad \text{and} \quad \lim_{\theta \to \frac{\pi}{4}^+} \theta \sec \theta = -\infty
\]

are not equal, the two-sided limit

\[
\lim_{\theta \to \frac{\pi}{4}} \theta \sec \theta \quad \text{does not exist.}
\]

38. \( \lim_{y \to 2} \left( \sin \frac{\pi}{y} \right) \)

**SOLUTION**

\[
\lim_{y \to 2} \left( \sin \frac{\pi}{y} \right) = \ln \left( \sin \frac{\pi}{2} \right) = \ln 1 = 0.
\]

39. \( \lim_{\theta \to 0} \frac{\cos \theta - 2}{\theta} \)

**SOLUTION**

Because the one-sided limits

\[
\lim_{\theta \to 0^-} \frac{\cos \theta - 2}{\theta} = \infty \quad \text{and} \quad \lim_{\theta \to 0^+} \frac{\cos \theta - 2}{\theta} = -\infty
\]

are not equal, the two-sided limit

\[
\lim_{\theta \to 0} \frac{\cos \theta - 2}{\theta} \quad \text{does not exist.}
\]
40. \[ \lim_{x \to 4.3} \frac{1}{x - [x]} \]
SOLUTION \[ \lim_{x \to 4.3} \frac{1}{x - [x]} = \frac{1}{4.3 - 4.3} = \frac{1}{0.3} = \frac{10}{3}. \]

41. \[ \lim_{x \to 2} \frac{x - 3}{x - 2} \]
SOLUTION For \( x \) close to 2 but less than 2, \( x - 3 < 0 \) and \( x - 2 < 0 \). Therefore,
\[ \lim_{x \to 2^{-}} \frac{x - 3}{x - 2} = \infty. \]

42. \[ \lim_{t \to 0} \frac{\sin^2 t}{t^5} \]
SOLUTION Note that
\[ \frac{\sin^2 t}{t^3} = \frac{\sin t}{t} \cdot \frac{\sin t}{t} \cdot \frac{1}{t}. \]
As \( t \to 0 \), each factor of \( \frac{\sin t}{t} \) approaches 1; however, the factor \( \frac{1}{t} \) tends to \( -\infty \) as \( t \to 0^- \) and tends to \( \infty \) as \( t \to 0^+ \). Consequently,
\[ \lim_{t \to 0^-} \frac{\sin^2 t}{t^3} = -\infty, \quad \lim_{t \to 0^+} \frac{\sin^2 t}{t^3} = \infty \]
and
\[ \lim_{t \to 0} \frac{\sin^2 t}{t^3} \] does not exist.

43. \[ \lim_{x \to 1^+} \left( \frac{1}{\sqrt{x - 1}} - \frac{1}{\sqrt{x^2 - 1}} \right) \]
SOLUTION \[ \lim_{x \to 1^+} \left( \frac{1}{\sqrt{x - 1}} - \frac{1}{\sqrt{x^2 - 1}} \right) = \lim_{x \to 1^+} \frac{\sqrt{x + 1} - 1}{\sqrt{x^2 - 1}} = \infty. \]

44. \[ \lim_{t \to e} \sqrt[t]{(\ln t - 1)} \]
SOLUTION
\[ \lim_{t \to e} \sqrt[t]{(\ln t - 1)} = \lim_{t \to e} \sqrt[t]{\ln t} \cdot \lim_{t \to e} (\ln t - 1) = \sqrt[e]{(\ln e - 1)} = 0. \]

45. \[ \lim_{x \to \frac{\pi}{2}} \tan x \]
SOLUTION Because the one-sided limits
\[ \lim_{x \to \frac{\pi}{2}^{-}} \tan x = \infty \quad \text{and} \quad \lim_{x \to \frac{\pi}{2}^{+}} \tan x = -\infty \]
are not equal, the two-sided limit
\[ \lim_{x \to \frac{\pi}{2}} \tan x \] does not exist.

46. \[ \lim_{t \to 0} \frac{1}{t} \]
SOLUTION As \( t \to 0 \), \( \frac{1}{t} \) grows without bound and \( \cos \left( \frac{1}{t} \right) \) oscillates faster and faster. Consequently,
\[ \lim_{t \to 0} \cos \left( \frac{1}{t} \right) \] does not exist.

The same is true for both one-sided limits.

47. \[ \lim_{t \to 0^+} \sqrt[t]{\cos \frac{1}{t}} \]
CHAPTER 2 | LIMITS

SOLUTION For \( t > 0 \),

\[-1 \leq \cos \left( \frac{1}{t} \right) \leq 1,\]

so

\[-\sqrt{t} \leq \sqrt{t} \cos \left( \frac{1}{t} \right) \leq \sqrt{t}.\]

Because

\[
\lim_{t \to 0^+} -\sqrt{t} = \lim_{t \to 0^+} \sqrt{t} = 0.
\]

it follows from the Squeeze Theorem that

\[
\lim_{t \to 0^+} \sqrt{t} \cos \left( \frac{1}{t} \right) = 0.
\]

48. \( \lim_{x \to 5^+} \frac{x^2 - 24}{x^2 - 25} \)

SOLUTION For \( x \) close to 5 but larger than 5, \( x^2 - 24 > 0 \) and \( x^2 - 25 > 0 \). Therefore,

\[
\lim_{x \to 5^+} \frac{x^2 - 24}{x^2 - 25} = \infty.
\]

49. \( \lim_{x \to 0} \frac{\cos x - 1}{\sin x} \)

SOLUTION

\[
\lim_{x \to 0} \frac{\cos x - 1}{\sin x} = \lim_{x \to 0} \frac{\cos x - 1}{\sin x} = \lim_{x \to 0} \frac{-\sin^2 x}{\cos x + 1} = \lim_{x \to 0} \frac{-\sin x}{\cos x + 1} = 0.
\]

50. \( \lim_{\theta \to 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} \)

SOLUTION

\[
\lim_{\theta \to 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} = \lim_{\theta \to 0} \frac{\sec \theta - 1}{\sin^2 \theta} = \lim_{\theta \to 0} \frac{\sec \theta - 1}{\sin^2 \theta} = \lim_{\theta \to 0} \frac{\sec \theta + 1}{\sin^2 \theta (\sec \theta + 1)} = \lim_{\theta \to 0} \frac{\sec^2 \theta}{\sec \theta + 1} = \frac{1}{1 + 1} = \frac{1}{2}.
\]

51. Find the left- and right-hand limits of the function \( f(x) \) in Figure 2 at \( x = 0, 2, 4 \). State whether \( f(x) \) is left- or right-continuous (or both) at these points.

![Figure 2](image)

SOLUTION According to the graph of \( f(x) \),

\[
\begin{align*}
\lim_{x \to 0^-} f(x) &= \lim_{x \to 0^+} f(x) = 1 \\
\lim_{x \to 2^-} f(x) &= \lim_{x \to 2^+} f(x) = \infty \\
\lim_{x \to 4^-} f(x) &= -\infty \\
\lim_{x \to 4^+} f(x) &= \infty.
\end{align*}
\]

The function is both left- and right-continuous at \( x = 0 \) and neither left- nor right-continuous at \( x = 2 \) and \( x = 4 \).
52. Sketch the graph of a function \( f(x) \) such that
(a) \( \lim_{x \to 2^-} f(x) = 1, \quad \lim_{x \to 2^+} f(x) = 3 \)
(b) \( \lim_{x \to 4} f(x) \) exists but does not equal \( f(4) \).

**SOLUTION**

\[ y \]

\[ x \]

53. Graph \( h(x) \) and describe the discontinuity:
\[ h(x) = \begin{cases} 
  e^x & \text{for } x \leq 0 \\
  \ln x & \text{for } x > 0
\end{cases} \]

Is \( h(x) \) left- or right-continuous?

**SOLUTION** The graph of \( h(x) \) is shown below. At \( x = 0 \), the function has an infinite discontinuity but is left-continuous.

54. Sketch the graph of a function \( g(x) \) such that
\[ \lim_{x \to 3^-} g(x) = \infty, \quad \lim_{x \to 3^+} g(x) = -\infty, \quad \lim_{x \to 4} g(x) = \infty \]

**SOLUTION**

55. Find the points of discontinuity of
\[ g(x) = \begin{cases} 
  \cos \left( \frac{\pi x}{2} \right) & \text{for } |x| < 1 \\
  |x - 1| & \text{for } |x| \geq 1
\end{cases} \]

Determine the type of discontinuity and whether \( g(x) \) is left- or right-continuous.

**SOLUTION** First note that \( \cos \left( \frac{\pi x}{2} \right) \) is continuous for \(-1 < x < 1\) and that \(|x - 1|\) is continuous for \( x \leq -1 \) and for \( x \geq 1 \). Thus, the only points at which \( g(x) \) might be discontinuous are \( x = \pm 1 \). At \( x = 1 \), we have
\[ \lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \cos \left( \frac{\pi x}{2} \right) = \cos \left( \frac{\pi}{2} \right) = 0 \]
and
\[ \lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} |x - 1| = |1 - 1| = 0. \]
so \( g(x) \) is continuous at \( x = 1 \). On the other hand, at \( x = -1 \),

\[
\lim_{x \to -1^+} g(x) = \lim_{x \to -1^+} \cos \left( \frac{\pi x}{2} \right) = \cos \left( \frac{-\pi}{2} \right) = 0
\]

and

\[
\lim_{x \to -1^-} g(x) = \lim_{x \to -1^-} |x - 1| = |1 - 1| = 2.
\]

so \( g(x) \) has a jump discontinuity at \( x = -1 \). Since \( g(-1) = 2 \), \( g(x) \) is left-continuous at \( x = -1 \).

56. Show that \( f(x) = xe^{\sin x} \) is continuous on its domain.

**SOLUTION** Because \( e^x \) and \( \sin x \) are continuous for all real numbers, their composition, \( e^{\sin x} \) is continuous for all real numbers. Moreover, \( x \) is continuous for all real numbers, so the product \( xe^{\sin x} \) is continuous for all real numbers. Thus, \( f(x) = xe^{\sin x} \) is continuous for all real numbers.

57. Find a constant \( b \) such that \( h(x) \) is continuous at \( x = 2 \), where

\[
h(x) = \begin{cases} 
  x + 1 & \text{for } |x| < 2 \\
  b - x^2 & \text{for } |x| \geq 2
\end{cases}
\]

With this choice of \( b \), find all points of discontinuity.

**SOLUTION** To make \( h(x) \) continuous at \( x = 2 \), we must have the two one-sided limits as \( x \) approaches 2 be equal. With

\[
\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} (x + 1) = 2 + 1 = 3
\]

and

\[
\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} (b - x^2) = b - 4,
\]

it follows that we must choose \( b = 7 \). Because \( x + 1 \) is continuous for \(-2 < x < 2 \) and \( 7 - x^2 \) is continuous for \( x \leq -2 \) and for \( x \geq 2 \), the only possible point of discontinuity is \( x = -2 \). At \( x = -2 \),

\[
\lim_{x \to -2^+} h(x) = \lim_{x \to 2^-} (x + 1) = -2 + 1 = -1
\]

and

\[
\lim_{x \to -2^-} h(x) = \lim_{x \to -2^-} (7 - x^2) = 7 - (-2)^2 = 3,
\]

so \( h(x) \) has a jump discontinuity at \( x = -2 \).

In Exercises 58–63, find the horizontal asymptotes of the function by computing the limits at infinity.

58. \( f(x) = \frac{9x^2 - 4}{2x^2 - x} \)

**SOLUTION** Because

\[
\lim_{x \to \infty} \frac{9x^2 - 4}{2x^2 - x} = \lim_{x \to \infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2}
\]

and

\[
\lim_{x \to -\infty} \frac{9x^2 - 4}{2x^2 - x} = \lim_{x \to -\infty} \frac{9 - 4/x^2}{2 - 1/x} = \frac{9}{2},
\]

it follows that the graph of \( y = \frac{9x^2 - 4}{2x^2 - x} \) has a horizontal asymptote of \( \frac{9}{2} \).

59. \( f(x) = \frac{x^2 - 3x^4}{x - 1} \)

**SOLUTION** Because

\[
\lim_{x \to \infty} \frac{x^2 - 3x^4}{x - 1} = \lim_{x \to \infty} \frac{1/x^2 - 3}{1/x - 1/x^4} = -\infty
\]

and

\[
\lim_{x \to -\infty} \frac{x^2 - 3x^4}{x - 1} = \lim_{x \to -\infty} \frac{1/x^2 - 3}{1/x - 1/x^4} = \infty,
\]

it follows that the graph of \( y = \frac{x^2 - 3x^4}{x - 1} \) does not have any horizontal asymptotes.
60. \( f(u) = \frac{8u - 3}{\sqrt{16u^2 + 6}} \)

**Solution** Because

\[
\lim_{u \to \infty} \frac{8u - 3}{\sqrt{16u^2 + 6}} = \lim_{u \to \infty} \frac{8 - 3/u}{\sqrt{16 + 6/u^2}} = \frac{8}{\sqrt{16}} = 2
\]

and

\[
\lim_{u \to -\infty} \frac{8u - 3}{\sqrt{16u^2 + 6}} = \lim_{u \to -\infty} \frac{8 - 3/u}{\sqrt{16 + 6/u^2}} = \frac{8}{-\sqrt{16}} = -2,
\]

it follows that the graph of \( y = \frac{8u - 3}{\sqrt{16u^2 + 6}} \) has horizontal asymptotes of \( y = \pm 2 \).

61. \( f(u) = \frac{2u^2 - 1}{\sqrt{6 + u^4}} \)

**Solution** Because

\[
\lim_{u \to \infty} \frac{2u^2 - 1}{\sqrt{6 + u^4}} = \lim_{u \to \infty} \frac{2 - 1/u^2}{\sqrt{6/u^4 + 1}} = \frac{2}{\sqrt{1}} = 2
\]

and

\[
\lim_{u \to -\infty} \frac{2u^2 - 1}{\sqrt{6 + u^4}} = \lim_{u \to -\infty} \frac{2 - 1/u^2}{\sqrt{6/u^4 + 1}} = \frac{2}{\sqrt{1}} = 2,
\]

it follows that the graph of \( y = \frac{2u^2 - 1}{\sqrt{6 + u^4}} \) has a horizontal asymptote of \( y = 2 \).

62. \( f(x) = \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} \)

**Solution** Because

\[
\lim_{x \to \infty} \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} = \lim_{x \to \infty} \frac{3x^{-2/15} + 9x^{-13/35}}{7 - x^{-17/15}} = 0
\]

and

\[
\lim_{x \to -\infty} \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} = \lim_{x \to -\infty} \frac{3x^{-2/15} + 9x^{-13/35}}{7 - x^{-17/15}} = 0,
\]

it follows that the graph of \( y = \frac{3x^{2/3} + 9x^{3/7}}{7x^{4/5} - 4x^{-1/3}} \) has a horizontal asymptote of \( y = 0 \).

63. \( f(t) = \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} \)

**Solution** Because

\[
\lim_{t \to \infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t \to \infty} \frac{1 - t^{2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1
\]

and

\[
\lim_{t \to -\infty} \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} = \lim_{t \to -\infty} \frac{1 - t^{2/3}}{(1 - t^{-2})^{1/3}} = \frac{1}{1^{1/3}} = 1,
\]

it follows that the graph of \( y = \frac{t^{1/3} - t^{-1/3}}{(t - t^{-1})^{1/3}} \) has a horizontal asymptote of \( y = 1 \).

64. Calculate (a)–(d), assuming that

\[
\lim_{x \to 3} f(x) = 6, \quad \lim_{x \to 3} g(x) = 4
\]

(a) \( \lim_{x \to 3} (f(x) - 2g(x)) \)

(b) \( \lim_{x \to 3} x^2 f(x) \)

(c) \( \lim_{x \to 3} \frac{f(x)}{g(x) + x} \)

(d) \( \lim_{x \to 3} (2g(x)^3 - g(x)^{3/2}) \)
**SOLUTION**

(a) \( \lim_{x \to 3} (f(x) - 2g(x)) = \lim_{x \to 3} f(x) - 2 \lim_{x \to 3} g(x) = 6 - 2(4) = -2. \)

(b) \( \lim_{x \to 3} x^2 f(x) = \lim_{x \to 3} x^2 \cdot \lim_{x \to 3} f(x) = 3^2 \cdot 6 = 54. \)

(c) \( \lim_{x \to 3} \frac{f(x)}{g(x) + x} = \lim_{x \to 3} \frac{f(x)}{g(x) + x} = \frac{6}{\lim_{x \to 3} g(x) + \lim_{x \to 3} x} = \frac{6}{4 + 3} = \frac{6}{7}. \)

(d) \( \lim_{x \to 3} (2g(x)^3 - g(x)^{3/2}) = \left( \lim_{x \to 3} g(x) \right)^3 - \left( \lim_{x \to 3} g(x) \right)^{3/2} = (2)^3 - (2)^{3/2} = 8 - \sqrt{8} = 8 - 2\sqrt{2}. \)

65. Assume that the following limits exist:

\[ A = \lim_{x \to a} f(x), \quad B = \lim_{x \to a} g(x), \quad L = \lim_{x \to a} \frac{f(x)}{g(x)} \]

Prove that if \( L = 1, \) then \( A = B. \) *Hint:* You cannot use the Quotient Law if \( B = 0, \) so apply the Product Law to \( L \) and \( B \) instead.

**SOLUTION** Suppose the limits \( A, B, \) and \( L \) all exist and \( L = 1. \) Then

\[ B = B \cdot L = \lim_{x \to a} g(x) \cdot \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} g(x) \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) = A. \]

66. Define \( g(t) = (1 + 2^{1/t})^{-1} \) for \( t \neq 0. \) How should \( g(0) \) be defined to make \( g(t) \) left-continuous at \( t = 0? \)

**SOLUTION** Because

\[ \lim_{t \to 0^-} (1 + 2^{1/t})^{-1} = \left[ \lim_{t \to 0^-} (1 + 2^{1/t}) \right]^{-1} = 1^{-1} = 1, \]

we should define \( g(0) = 1 \) to make \( g(t) \) left-continuous at \( t = 0. \)

67. In the notation of Exercise 65, give an example where \( L \) exists but neither \( A \) nor \( B \) exists.

**SOLUTION** Suppose

\[ f(x) = \frac{1}{(x-a)^3}, \quad g(x) = \frac{1}{(x-a)^5}. \]

Then, neither \( A \) nor \( B \) exists, but

\[ L = \lim_{x \to a} \frac{(x-a)^3}{(x-a)^5} = \lim_{x \to a} (x-a)^{-2} = 0. \]

68. True or false?

(a) If \( \lim_{x \to 3} f(x) \) exists, then \( \lim_{x \to 3} f(x) = f(3). \)

(b) If \( \lim_{x \to 0} \frac{f(x)}{x} = 1, \) then \( f(0) = 0. \)

(c) If \( \lim_{x \to 7} f(x) = 8, \) then \( \lim_{x \to 7} \frac{1}{f(x)} = \frac{1}{8}. \)

(d) If \( \lim_{x \to 5^+} f(x) = 4 \) and \( \lim_{x \to 5^-} f(x) = 8, \) then \( \lim_{x \to 5} f(x) = 6. \)

(e) If \( \lim_{x \to 0} \frac{f(x)}{x} = 1, \) then \( \lim_{x \to 0} f(x) = 0. \)

(f) If \( \lim_{x \to 5} f(x) = 2, \) then \( \lim_{x \to 5} f(x)^3 = 8. \)

**SOLUTION**

(a) False. The limit \( \lim_{x \to 3} f(x) \) may exist and need not equal \( f(3). \) The limit is equal to \( f(3) \) if \( f(x) \) is continuous at \( x = 3. \)

(b) False. The value of the limit \( \lim_{x \to 0} \frac{f(x)}{x} \) does not depend on the value \( f(0), \) so \( f(0) \) can have any value.

(c) True, by the Limit Laws.

(d) False. If the two one-sided limits are not equal, then the two-sided limit does not exist.

(e) True. Apply the Product Law to the functions \( \frac{f(x)}{x} \) and \( x. \)

(f) True, by the Limit Laws.

69. Let \( f(x) = x \left[ \frac{1}{x} \right], \) where \( \left[ x \right] \) is the greatest integer function. Show that for \( x \neq 0, \)

\[ \frac{1}{x} - 1 < \left[ \frac{1}{x} \right] \leq \frac{1}{x} \]

Then use the Squeeze Theorem to prove that

\[ \lim_{x \to 0} x \left[ \frac{1}{x} \right] = 1 \]

*Hint:* Treat the one-sided limits separately.
SOLUTION Let $y$ be any real number. From the definition of the greatest integer function, it follows that $y - 1 < \lceil y \rceil \leq y$, with equality holding if and only if $y$ is an integer. If $x \neq 0$, then $\frac{1}{x}$ is a real number, so

$$ \frac{1}{x} - 1 < \left\lfloor \frac{1}{x} \right\rfloor \leq \frac{1}{x}. $$

Upon multiplying this inequality through by $x$, we find

$$ 1 - x < x \left\lfloor \frac{1}{x} \right\rfloor \leq 1. $$

Because

$$ \lim_{x \to 0} (1 - x) = \lim_{x \to 0} 1 = 1, $$

it follows from the Squeeze Theorem that

$$ \lim_{x \to 0} x \left\lfloor \frac{1}{x} \right\rfloor = 1. $$

70. Let $r_1$ and $r_2$ be the roots of $f(x) = ax^2 - 2x + 20$. Observe that $f(x)$ “approaches” the linear function $L(x) = -2x + 20$ as $a \to 0$. Because $r = 10$ is the unique root of $L(x)$, we might expect one of the roots of $f(x)$ to approach 10 as $a \to 0$ (Figure 3). Prove that the roots can be labeled so that $\lim_{a \to 0} r_1 = 10$ and $\lim_{a \to 0} r_2 = \infty$.

![Figure 3: Graphs of $f(x) = ax^2 - 2x + 20$.](image)

SOLUTION Using the quadratic formula, we find that the roots of the quadratic polynomial $f(x) = ax^2 - 2x + 20$ are

$$ r_1 = \frac{20}{1 + \sqrt{1 - 20a}}, \quad r_2 = \frac{20}{1 - \sqrt{1 - 20a}}. $$

Now let

$$ r_1 = \frac{20}{1 + \sqrt{1 - 20a}} \quad \text{and} \quad r_2 = \frac{20}{1 - \sqrt{1 - 20a}}. $$

It is straightforward to calculate that

$$ \lim_{a \to 0} r_1 = \lim_{a \to 0} \frac{20}{1 + \sqrt{1 - 20a}} = \frac{20}{2} = 10 $$

and that

$$ \lim_{a \to 0} r_2 = \lim_{a \to 0} \frac{20}{1 - \sqrt{1 - 20a}} = \infty $$

as desired.

71. Use the IVT to prove that the curves $y = x^2$ and $y = \cos x$ intersect.

SOLUTION Let $f(x) = x^2 - \cos x$. Note that any root of $f(x)$ corresponds to a point of intersection between the curves $y = x^2$ and $y = \cos x$. Now, $f(x)$ is continuous over the interval $[0, \frac{\pi}{2}]$, $f(0) = -1 < 0$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} > 0$. Therefore, by the Intermediate Value Theorem, there exists a $c \in (0, \frac{\pi}{2})$ such that $f(c) = 0$; consequently, the curves $y = x^2$ and $y = \cos x$ intersect.

72. Use the IVT to prove that $f(x) = x^3 - \frac{x^2 + 2}{\cos x + 2}$ has a root in the interval $[0, 2]$.

SOLUTION Let $f(x) = x^3 - \frac{x^2 + 2}{\cos x + 2}$. Because $\cos x + 2$ is never zero, $f(x)$ is continuous for all real numbers. Because

$$ f(0) = -2 < 0 \quad \text{and} \quad f(2) = 8 - \frac{6}{\cos 2 + 2} \approx 4.21 > 0, $$

the Intermediate Value Theorem guarantees there exists a $c \in (0, 2)$ such that $f(c) = 0$. 

73. Use the IVT to show that \( e^{-x^2} = x \) has a solution on \((0, 1)\).

**SOLUTION**  Let \( f(x) = e^{-x^2} - x \). Observe that \( f \) is continuous on \([0, 1]\) with \( f(0) = e^0 - 0 = 1 > 0 \) and \( f(1) = e^{-1} - 1 < 0 \). Therefore, the IVT guarantees there exists a \( c \in (0, 1) \) such that \( f(c) = e^{-c^2} - c = 0 \).

74. Use the Bisection Method to locate a solution of \( x^2 - 7 = 0 \) to two decimal places.

**SOLUTION**  Let \( f(x) = x^2 - 7 \). By trial and error, we find that \( f(2.6) = -0.24 < 0 \) and \( f(2.7) = 0.29 > 0 \). Because \( f(x) \) is continuous on \([2.6, 2.7]\), it follows from the Intermediate Value Theorem that \( f(x) \) has a root on \((2.6, 2.7)\). We approximate the root by the midpoint of the interval: \( x = 2.65 \). Now, \( f(2.65) = 0.0225 > 0 \). Because \( f(2.6) \) and \( f(2.65) \) are of opposite sign, the root must lie on \((2.6, 2.65)\). The midpoint of this interval is \( x = 2.625 \) and \( f(2.625) < 0 \); hence, the root must be on the interval \((2.625, 2.65)\). Continuing in this fashion, we construct the following sequence of intervals and midpoints.

<table>
<thead>
<tr>
<th>interval</th>
<th>midpoint</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.625, 2.65)</td>
<td>2.6375</td>
</tr>
<tr>
<td>(2.6375, 2.65)</td>
<td>2.64375</td>
</tr>
<tr>
<td>(2.64375, 2.65)</td>
<td>2.646875</td>
</tr>
<tr>
<td>(2.64375, 2.646875)</td>
<td>2.6453125</td>
</tr>
<tr>
<td>(2.6453125, 2.646875)</td>
<td>2.64609375</td>
</tr>
</tbody>
</table>

At this point, we note that, to two decimal places, one root of \( x^2 - 7 = 0 \) is 2.65.

75. Give an example of a (discontinuous) function that does not satisfy the conclusion of the IVT on \([-1, 1]\). Then show that the function

\[
f(x) = \begin{cases} 
\sin \frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0 
\end{cases}
\]

satisfies the conclusion of the IVT on every interval \([-a, a]\), even though \( f \) is discontinuous at \( x = 0 \).

**SOLUTION**  Let \( g(x) = |x| \). This function is discontinuous on \([-1, 1]\) with \( g(-1) = -1 \) and \( g(1) = 1 \). For all \( c \neq 0 \), there is no \( x \) such that \( g(x) = c \); thus, \( g(x) \) does not satisfy the conclusion of the Intermediate Value Theorem on \([-1, 1]\).

Now, let

\[
f(x) = \begin{cases} 
\sin \left( \frac{1}{x} \right) & \text{for } x \neq 0 \\
0 & \text{for } x = 0 
\end{cases}
\]

and let \( a > 0 \). On the interval

\[
x \in \left[ -\frac{a}{2+2\pi a}, \frac{a}{2} \right] \subseteq [-a, a],
\]

\( \frac{1}{x} \) runs from \( \frac{a}{2} \) to \( \frac{a}{2} + 2\pi \), so the sine function covers one full period and clearly takes on every value from \(-\sin a\) through \(\sin a\).

76. Let \( f(x) = \frac{1}{x+2} \).

(a) Show that \( |f(x) - \frac{1}{4}| < \frac{|x-2|}{12} \) if \( |x-2| < 1 \). *Hint: Observe that \(|4(x+2)| > 12 \) if \( |x-2| < 1 \).*

(b) Find \( \delta > 0 \) such that \( |f(x) - \frac{1}{4}| < 0.01 \) for \( |x-2| < \delta \).

(c) Prove rigorously that \( \lim_{x \to 2} f(x) = \frac{1}{4} \).

**SOLUTION**

(a) Let \( f(x) = \frac{1}{x+2} \). Then

\[
|f(x) - \frac{1}{4}| = \left| \frac{1}{x+2} - \frac{1}{4} \right| = \frac{|4 - (x+2)|}{4(x+2)} = \frac{|x-2|}{|4(x+2)|}.
\]

If \( |x-2| < 1 \), then \( 1 < x < 3 \), so \( 3 < x+2 < 5 \) and \( 12 < 4(x+2) < 20 \). Hence,

\[
\frac{1}{|4(x+2)|} < \frac{1}{12} \quad \text{and} \quad |f(x) - \frac{1}{4}| < \frac{|x-2|}{12}.
\]

(b) If \( |x-2| < \delta \), then by part (a),

\[
|f(x) - \frac{1}{4}| < \frac{\delta}{12}.
\]

Choosing \( \delta = 0.12 \) will then guarantee that \( |f(x) - \frac{1}{4}| < 0.01 \).
(c) Let $\epsilon > 0$ and take $\delta = \min\{1, 12\epsilon\}$. Then, whenever $|x - 2| < \delta$,

$$|f(x) - \frac{1}{4}| = \frac{1}{|x + 2|} \frac{1}{4} \frac{|2 - x|}{|x + 2|} \frac{1}{\frac{x - 2}{12}} < \frac{\delta}{12} = \epsilon.$$  

77. [GU] Plot the function $f(x) = x^{1/3}$. Use the zoom feature to find a $\delta > 0$ such that $|x^{1/3} - 2| < 0.05$ for $|x - 8| < \delta$.

**SOLUTION** The graphs of $y = f(x) = x^{1/3}$ and the horizontal lines $y = 1.95$ and $y = 2.05$ are shown below. From this plot, we see that $\delta = 0.55$ guarantees that $|x^{1/3} - 2| < 0.05$ whenever $|x - 8| < \delta$.

![Graph of $x^{1/3}$](image)

78. Use the fact that $f(x) = 2^x$ is increasing to find a value of $\delta$ such that $|2^x - 8| < 0.001$ if $|x - 2| < \delta$. Hint: Find $c_1$ and $c_2$ such that $7.999 < f(c_1) < f(c_2) < 8.001$.

**SOLUTION** From the graph below, we see that $7.999 < f(2.99985) < f(3.00015) < 8.001$.

Thus, with $\delta = 0.00015$, it follows that $|2^x - 8| < 0.001$ if $|x - 3| < \delta$.

![Graph of $2^x$](image)

79. Prove rigorously that $\lim_{x \to -1} (4 + 8x) = -4$.

**SOLUTION** Let $\epsilon > 0$ and take $\delta = \epsilon/8$. Then, whenever $|x - (-1)| = |x + 1| < \delta$,

$$|f(x) - (-4)| = |4 + 8x + 4| = 8|x + 1| < 8\delta \leq \epsilon.$$  

80. Prove rigorously that $\lim_{x \to 3} (x^2 - x) = 6$.

**SOLUTION** Let $\epsilon > 0$ and take $\delta = \min\{1, \epsilon/6\}$. Because $\delta \leq 1$, $|x - 3| < \delta$ guarantees $|x + 2| < 6$. Therefore, whenever $|x - 3| < \delta$,

$$|f(x) - 6| = |x^2 - x - 6| = |x - 3||x + 2| < 6|x - 3| < 6\delta \leq \epsilon.$$