Chapter 2: Limits
Preparing for the AP Exam Solutions

Multiple Choice Questions


Free Response Questions

1. a) 
\[
\frac{f\left(\frac{3\pi}{2}\right) - f\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2} - \frac{\pi}{2}} = \frac{-1 - \frac{1}{\pi}}{\frac{2\pi}{2}} = -1 \left(\frac{\pi}{3\pi} + \frac{2}{\pi}\right) = \frac{-8}{3\pi^2}
\]

b) \[\lim_{x \to 0} f(x) = 1\]

c) No, \[\lim_{x \to 0} f(x) = 1\], so neither the left-hand limit nor the right hand limit is infinite, which is needed for the graph to have a vertical asymptote.

d) We know \(-1 \leq \sin x \leq 1\), so if \(x > 0\), then \[-\frac{1}{x} \leq \frac{\sin x}{x} \leq 1\], and since \[\lim_{x \to \infty} -\frac{1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}\], the Squeeze Theorem implies \[\lim_{x \to \infty} \frac{\sin x}{x} = 0\]. This means the line \(y = 0\) is a horizontal asymptote.

POINTS:
(a) (2 pts) 1) change in \(y\); 1) answer  
(b) (1 pt)
(c) (3 pts) 1) “no”; 1) mentioning finite limit; 1) mentioning need for infinite limit  
(d) (3 pts) 1) \[-1 \leq \frac{\sin x}{x} \leq 1\]; 1) \[\lim_{x \to \infty} -\frac{1}{x} = 0 = \lim_{x \to \infty} \frac{1}{x}\]; 1) conclusion

2. a) The function \(f(x) = \frac{x^2 - 7x + 10}{x^2 - 25}\) is discontinuous at \(x = 5\) and \(x = -5\). First, \[\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 25} = \frac{3}{10}\]. Thus the line \(x = 5\) is not a vertical asymptote. Next, 
\[
\lim_{x \to -5} \frac{x^2 - 7x + 10}{x^2 - 25} = \lim_{x \to -5} \frac{x - 2}{x + 5} = -\infty
\]
Thus the line \(x = -5\) is a vertical asymptote.

b) \[\lim_{x \to \infty} \frac{x^2 - 7x + 10}{x^2 - 25} = 1\], so the line \(y = 1\) is a horizontal asymptote. Also \[\lim_{x \to \infty} \frac{3}{10}\], the line \(y = 1\) is the only horizontal asymptote.

c) Yes, since \[\lim_{x \to 5} f(x) = \frac{3}{10}\], we can let \(A = \frac{3}{10}\).
d) No, since \( \lim_{x \to 5} f(x) \) does not exist, there is no possible value for \( B \).

POINTS:
(a) (4 pts) 1) “no” for \( x = 5 \); 1) Limit is \( \frac{3}{10} \); 1) “yes” for \( x = -5 \); 1) infinite limit
(b) (3 pts) 1) \( y = 1 \); 1) Limit at \( \infty \). 1) Limit at \( -\infty \).
(c) (1pt) \( A = \frac{3}{10} \)
(d) (1 pt) No limit.

3. a) Since \(-5 \leq f(x) \leq 10\), if \( x > 0 \) then \(-5x \leq xf(x) \leq 10x\). Thus by the Squeeze Theorem \( \lim_{x \to 0^+} xf(x) = 0 \). Next, if \( x < 0 \), then \(-5x \geq xf(x) \geq 10x\). Applying the Squeeze Theorem again, \( \lim_{x \to 0^-} xf(x) = 0 \). Thus \( \lim_{x \to 0} xf(x) = \lim_{x \to 0} g(x) = 0 \). Checking the functional value, we have \( g(0) = 0 \cdot 3 = 0 \). Thus \( \lim_{x \to 0} g(x) = g(0) \), so \( g \) is continuous at \( x = 0 \).

b) No. \( \lim_{x \to 0} \frac{g(x) - 0}{x - 0} = \lim_{x \to 0} \frac{xf(x)}{x} = \lim_{x \to 0} f(x) \), which does not exist.

POINTS:
(a) (6 pts) 1) \( g(0) = 0 \); 1) if \( x > 0 \) then \(-5x \leq xf(x) \leq 10x\); 1) \( \lim_{x \to 0^+} xf(x) = 0 \);
if \( x < 0 \), then \(-5x \geq xf(x) \geq 10x\); 1) \( \lim_{x \to 0^-} xf(x) = 0 \); 1) \( \lim_{x \to 0} g(x) = 0 \)
(b) (3 pts) 1) Considers \( \lim_{x \to 0} \frac{g(x) - 0}{x - 0} \); 1) \( \lim_{x \to 0} \frac{g(x) - 0}{x - 0} = \lim_{x \to 0} f(x) \); 1) Answer

4. a) First, \( \lim_{x \to 4} f(x) = \lim_{x \to 4} (6 - x) = 2 \). Next \( \lim_{x \to 4} f(x) = \lim_{x \to 4} \sqrt[3]{2x} = \sqrt[3]{8} = 2 \).
So \( \lim_{x \to 4} f(x) = 2 \). Also \( f(4) = 2 \), which means \( f \) is continuous at \( x = 4 \).

b) \( \frac{f(0.004) - f(0)}{0.004 - 0} = \frac{0.008}{0.004} = \frac{2}{0.004} = 50 \)

c) No, \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{2x}}{x} = \lim_{x \to 0} \frac{\sqrt[3]{2}}{x^{2/3}} \) does not exist.

POINTS:
(a) (5 pts) 1) \( \lim_{x \to 4} (6 - x) = 2 \); 1) \( \lim_{x \to 4} \sqrt[3]{2x} = 2 \); 1) \( \lim_{x \to 4} f(x) = 2 \); 1) \( f(4) = 2 \); 1) Answer
(b) (1 pt)
(c) (3 pts) 1) Considers \( \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} \); 1) \( \lim_{x \to 0} \frac{\sqrt[3]{2x}}{x} \); 1) \( \lim_{x \to 0} \frac{\sqrt[3]{2}}{x^{2/3}} \) does not exist.