Chapter 4: Application of the Derivative
Preparing for the AP Exam Solutions

Multiple Choice Questions
1) B  2) E  3) B  4) A  5) C
6) A  7) C  8) B  9) D  10) B

Free Response Questions
1. a) No. There are various justifications. For example, \( \frac{dx}{dt} < 0 \) when \( y > 0 \) and \( \frac{dx}{dt} > 0 \) when \( y < 0 \) since the runner is going counterclockwise.

Or, \( \frac{dx}{dt} = 0 \) when \( y = 0 \), and since the runner is moving, \( \frac{dx}{dt} \) cannot be constantly zero.

b) Let \( P = (x, y) \) be a point in the first quadrant on the ellipse. Then construct the rectangle \( R \) with vertices \( (x, y), (-x, y), (x, -y), \) and \( (-x, -y). \) The area of \( R \) is \( A = 4xy. \) Since \( y > 0, \ y = \frac{\sqrt{50000 - 10x^2}}{2} \), so

\[
A = 2x\sqrt{50000 - 10x^2}. \quad A'(x) = \frac{100000 - 4x^2}{\sqrt{50000 - 10x^2}} = 0 \text{ when } x = 50 \text{ (remember } x > 0) \text{ and changes sign from plus to minus. Thus } A \text{ has a maximum at } x = 50. \ A(50) = A = 5000\sqrt{10}.

Next, \( A(1) < 5000 < A(50) \), so the Intermediate Value Theorem says there is a rectangle with area exactly 5000 square yards.

POINTS:
(a) (2 pts) 1) Answer 2) Justification
(b) (7pts) 1) \( A = 4xy \); 1) \( A = 2x\sqrt{50000 - 10x^2} \); 1) \( A'(x) = \frac{100000 - 4x^2}{\sqrt{50000 - 10x^2}} \); 1) \( x = 50 \); 1) justifies maximum at \( x = 50 \); 1) \( A(50) > 5000 \); 1) uses IVT

2. a) No. \( f''(x) \) is continuous for all \( x \), thus \( f(x) \) is continuous for all \( x \), which means there is not a vertical asymptote.

b) Yes. We have no information about \( \lim_{x \to \infty} f(x) \) or \( \lim_{x \to -\infty} f(x) \).

c) \[
\frac{f'(2) - f'(0)}{2 - 0} = -1. \quad \text{Apply the Mean Value Theorem to } f'(x) \text{ on } [0,2]. \text{ There is a } c \text{ in } (0, 2) \text{ with } f''(c) = -1 < 0
\]
d) Note that \( f'(2) = 0 \) and \( f''(2) = 1 > 0 \), so \( f(x) \) has a local minimum at \( x = 2 \). Thus \( f(x) \) is decreasing just to the left of \( x = 2 \), and so \( f'(x) < 0 \) just to the left of \( x = 2 \).

POINTS:
(a) (2 pts) 1) Answer; 1) Reason
(b) (2 pts) 1) Answer; 1) Reason
(c) (2 pts) 1) \( \frac{f'(2) - f'(0)}{2 - 0} = -1; \) 1) Uses MVT

(d) (3 pts) 1) local min at \( x = 2; \) 1) \( f(x) \) decreasing on some interval; 1) decreasing means \( f'(x) < 0. \)

3. a) If \( k = 30, \) then \( f(0) = 30. \) Since \( f(x) \) is a cubic and the coefficient of \( x^3 > 0, \) we know \( f(x) \) will be negative for some negative values of \( x. \) Experimenting, we find \( f(-3) = -15. \) Thus there is a \( c \) in \((-3, 0)\) with \( f(c) = 0. \)

b) We have \( f'(x) = 6x^2 - 6x - 12 = 6(x+1)(x-2) \) Thus \( f(x) \) has a local maximum \( x = -1 \) and a local minimum at \( x = 2. \) With \( k = 30, f(x) \) is increasing on \((-\infty, -1] \) and \( f(-1) = 37. \) So \( f(x) = 0 \) has exactly one solution in \((-\infty, -1]. \) \( f(x) \) is decreasing on \([-1, 2], \) and \( f(2) = 10, \) so there is no solution to \( f(x) = 0 \) in \([-1, 2]. \)

f(x) now is increasing for \( x > 2, \) so there is no solution to \( f(x) = 0 \) in \((2, \infty). \)

c) We want the graph to intersect the \( x- \) axis exactly once, so we want either (i) the local maximum to be less than 0 or (ii) the local minimum to be greater than 0. For (i), \( kf + 7 < 0 \) so \( k < -7. \) For (ii), \( k + 20 \) so \( k > 20. \) [Note, for \( k = -7 \) or 20, there are exactly two solutions.]

POINTS:
(a) (2 pts) 1) Finds a negative value of \( f(x) \) 1) Uses IVT
(b) (5 pts) 1) local max at \( x = -1; \) 1) local min at \( x = 2; \) 1) deals with \((-\infty, -1]; \) 1) deals with \([-1, 2]; \) 1) deals with \((2, \infty). \)
(c) (2 pts) 1) \( k < -7; \) 1) \( k > 20 \)

4. a) Differentiating with respect to \( x, \) we get \( 2x - y - x \frac{dy}{dx} + 2 \frac{dy}{dx} = 0. \) Thus \( \frac{dy}{dx} = \frac{y-2x}{2y-x}. \) At the point \((-2, 3), \) \( \frac{dy}{dx} = \frac{7}{8}. \) The equation is \( y = 3 + \frac{7}{8}(x+2). \)

b) \( q \approx 3 + \frac{7}{8}(-2.168 + 2) = 2.853 \)

c) Use our calculator to solve \((-2.168)^2 - (-2.168)y + y^2 = 19. \) Choose the solution nearest 3; to three decimal places \( y = 2.849. \) This is less than the tangent line approximation, so the graph is below the tangent line, and it appears the graph is concave down.

d) \( \frac{d^2y}{dx^2} \) so \( \frac{d^2y}{dx^2} = \frac{(\frac{dy}{dx} - 2)(2y-x) - (y-2x)(2 \frac{dy}{dx} - 1)}{(2y-x)^2}. \) Next substitute \( x = -2, y = 3 \) and \( \frac{dy}{dx} = \frac{7}{8} \) to get \( \frac{d^2y}{dx^2} = -\frac{57}{256} < 0, \) which confirms our answer of concave down in part (c).

POINTS:
(a) (3 pts) 1) \( 2x - y - x \frac{dy}{dx} + 2 \frac{dy}{dx} = 0; \) 1) \( \frac{dy}{dx} = \frac{7}{8}; \) 1) \( y = 3 + \frac{7}{8}(x+2) \)
(b) (1 pt)
(c) (2 pts) 1) \( y = 2.850; \) 2) conclusion of concave down
(d) (3 pts) 1) \( \frac{d^2y}{dx^2} = \frac{(\frac{dy}{dx} - 2)(2y-x) - (y-2x)(2 \frac{dy}{dx} - 1)}{(2y-x)^2}; \) 1) \( \frac{d^2y}{dx^2} = -\frac{57}{256} < 0; \)

1) Concludes concave down