87. Suppose that \( f \) and \( g \) are continuous functions such that, for all \( a \),

\[
\int_{-a}^{a} f(x) \, dx = \int_{-a}^{a} g(x) \, dx
\]

Give an intuitive argument showing that \( f(0) = g(0) \). Explain your idea with a graph.

**SOLUTION** Let \( \int_{-a}^{a} f(x) \, dx = \int_{-a}^{a} g(x) \, dx \). Consider what happens as \( a \) decreases in size, becoming very close to zero. Intuitively, the areas of the functions become \((a - (-a))(f(0)) = 2a(f(0))\) and \((a - (-a))(g(0)) = 2a(g(0))\). Because we know these areas must be the same, we have \(2a(f(0)) = 2a(g(0))\) and therefore \(f(0) = g(0)\).

88. Theorem 4 remains true without the assumption \( a \leq b \leq c \). Verify this for the cases \( b < a < c \) and \( c < a < b \).

**SOLUTION** The additivity property of definite integrals states for \( a \leq b \leq c \), we have \( \int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx \).

- Suppose that we have \( b < a < c \). By the additivity property, we have \( \int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx \). Therefore, \( \int_{a}^{a} f(x) \, dx = \int_{b}^{a} f(x) \, dx - \int_{c}^{a} f(x) \, dx = \int_{b}^{c} f(x) \, dx \).
- Now suppose that we have \( c < a < b \). By the additivity property, we have \( \int_{c}^{b} f(x) \, dx = \int_{c}^{a} f(x) \, dx + \int_{a}^{b} f(x) \, dx \). Therefore, \( \int_{a}^{c} f(x) \, dx = -\int_{c}^{a} f(x) \, dx = \int_{b}^{a} f(x) \, dx - \int_{a}^{b} f(x) \, dx = \int_{c}^{b} f(x) \, dx + \int_{a}^{b} f(x) \, dx \).
- Hence the additivity property holds for all real numbers \( a, b, \) and \( c \), regardless of their relationship amongst each other.

### 5.3 The Fundamental Theorem of Calculus, Part I

#### Preliminary Questions

1. Suppose that \( F'(x) = f(x) \) and \( F(0) = 3, F(2) = 7 \).
   (a) What is the area under \( y = f(x) \) over \([0, 2]\) if \( f(x) \geq 0 \)?
   (b) What is the graphical interpretation of \( F(2) - F(0) \) if \( f(x) \) takes on both positive and negative values?

**SOLUTION**

(a) If \( f(x) \geq 0 \) over \([0, 2]\), then the area under \( y = f(x) \) is \( F(2) - F(0) = 7 - 3 = 4 \).

(b) If \( f(x) \) takes on both positive and negative values, then \( F(2) - F(0) \) gives the signed area between \( y = f(x) \) and the \( x \)-axis.

2. Suppose that \( f(x) \) is a negative function with antiderivative \( F \) such that \( F(1) = 7 \) and \( F(3) = 4 \). What is the area (a positive number) between the \( x \)-axis and the graph of \( f(x) \) over \([1, 3]\)?

**SOLUTION** \( \int_{1}^{3} f(x) \, dx \) represents the signed area bounded by the curve and the interval \([1, 3]\). Since \( f(x) \) is negative on \([1, 3]\), \( \int_{1}^{3} f(x) \, dx \) is the negative of the area. Therefore, if \( A \) is the area between the \( x \)-axis and the graph of \( f(x) \), we have:

\[
A = -\int_{1}^{3} f(x) \, dx = -(F(3) - F(1)) = -(4 - 7) = -(-3) = 3.
\]

3. Are the following statements true or false? Explain.
   (a) FTC I is valid only for positive functions.
   (b) To use FTC I, you have to choose the right antiderivative.
   (c) If you cannot find an antiderivative of \( f(x) \), then the definite integral does not exist.

**SOLUTION**

(a) False. The FTC I is valid for continuous functions.

(b) False. The FTC I works for any antiderivative of the integrand.

(c) False. If you cannot find an antiderivative of the integrand, you cannot use the FTC I to evaluate the definite integral, but the definite integral may still exist.
4. Evaluate \( \int_2^9 f'(x) \, dx \) where \( f(x) \) is differentiable and \( f(2) = f(9) = 4 \).

**SOLUTION** Because \( f \) is differentiable, \( \int_2^9 f'(x) \, dx = f(9) - f(2) = 4 - 4 = 0 \).

**Exercises**

In Exercises 1–4, sketch the region under the graph of the function and find its area using FTC I.

1. \( f(x) = x^2, \quad [0, 1] \)

**SOLUTION**

![Graph](image1)

We have the area

\[
A = \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \bigg|_0^1 = \frac{1}{3}.
\]

2. \( f(x) = 2x - x^2, \quad [0, 2] \)

**SOLUTION**

![Graph](image2)

Let \( A \) be the area indicated. Then:

\[
A = \int_0^2 (2x - x^2) \, dx = \int_0^2 2x \, dx - \int_0^2 x^2 \, dx = x^2 \bigg|_0^2 - \frac{1}{3} x^3 \bigg|_0^2 = (4 - 0) - \left( \frac{8}{3} - 0 \right) = \frac{4}{3}.
\]

3. \( f(x) = x^{-2}, \quad [1, 2] \)

**SOLUTION**

![Graph](image3)

We have the area

\[
A = \int_1^2 x^{-2} \, dx = \frac{x^{-1}}{-1} \bigg|_1^2 = \frac{1}{2} + 1 = \frac{1}{2}.
\]

4. \( f(x) = \cos x, \quad [0, \frac{\pi}{2}] \)

**SOLUTION**
Let $A$ be the shaded area. Then

$$A = \int_0^{\pi/2} \cos x \, dx = \left[ \sin x \right]_0^{\pi/2} = 1 - 0 = 1.$$ 

In Exercises 5–42, evaluate the integral using FTC I.

5. $\int_3^6 x \, dx$

**Solution** $\int_3^6 x \, dx = \frac{1}{2} x^2 \bigg|_3^6 = \frac{1}{2} (6^2 - 3^2) = \frac{27}{2}$

6. $\int_0^9 2 \, dx$

**Solution** $\int_0^9 2 \, dx = 2x \bigg|_0^9 = 2(9) - 2(0) = 18.$

7. $\int_0^1 (4x - 9x^2) \, dx$

**Solution** $\int_0^1 (4x - 9x^2) \, dx = (2x^2 - 3x^3) \bigg|_0^1 = (2 - 3) - (0 - 0) = -1.$

8. $\int_{-3}^2 u^2 \, du$

**Solution** $\int_{-3}^2 u^2 \, du = \frac{1}{3} u^3 \bigg|_{-3}^2 = \frac{1}{3} (2)^3 - \frac{1}{3} (-3)^3 = \frac{35}{3}$.

9. $\int_0^2 (12x^5 + 3x^2 - 4x) \, dx$

**Solution** $\int_0^2 (12x^5 + 3x^2 - 4x) \, dx = (2x^6 + x^3 - 2x^2) \bigg|_0^2 = (128 + 8) - (0 + 0 - 0) = 128.$

10. $\int_{-2}^0 (10x^9 + 3x^5) \, dx$

**Solution** $\int_{-2}^0 (10x^9 + 3x^5) \, dx = (x^{10} + \frac{1}{2} x^6) \bigg|_{-2}^0 = \left( 2^{10} + \frac{1}{2} 2^6 \right) - \left( 2^{10} + \frac{1}{2} 2^6 \right) = 0.$

11. $\int_3^0 (2t^3 - 6t^2) \, dt$

**Solution** $\int_3^0 (2t^3 - 6t^2) \, dt = \left( \frac{1}{2} t^4 - 2t^3 \right) \bigg|_3^0 = (0 - 0) - \left( \frac{81}{2} - 54 \right) = \frac{27}{2}$.

12. $\int_{-1}^1 (5u^4 + u^2 - u) \, du$

**Solution** $\int_{-1}^1 (5u^4 + u^2 - u) \, du = \left( u^5 + \frac{1}{3} u^3 - \frac{1}{2} u^2 \right) \bigg|_{-1}^1 = \left( 1 + \frac{1}{3} - \frac{1}{2} \right) - \left( -1 - \frac{1}{3} - \frac{1}{2} \right) = \frac{8}{3}$.

13. $\int_0^4 \sqrt[3]{y} \, dy$

**Solution** $\int_0^4 \sqrt[3]{y} \, dy = \int_0^4 y^{1/2} \, dy = \frac{2}{3} y^{3/2} \bigg|_0^4 = \frac{2}{3} (4)^{3/2} - \frac{2}{3} (0)^{3/2} = \frac{16}{3}.$
14. \( \int_1^8 x^{4/3} \, dx \)  
SOLUTION \( \int_1^8 x^{4/3} \, dx = \frac{3}{7} x^{7/3} \bigg|_1^8 = \frac{3}{7} (128 - 1) = \frac{381}{7} \).

15. \( \int_{1/16}^1 t^{1/4} \, dt \)  
SOLUTION \( \int_{1/16}^1 t^{1/4} \, dt = \frac{4}{5} t^{5/4} \bigg|_{1/16}^1 = \frac{4}{5} - \frac{1}{40} = \frac{31}{40} \).

16. \( \int_4^1 t^{5/2} \, dt \)  
SOLUTION \( \int_4^1 t^{5/2} \, dt = \frac{2}{7} t^{7/2} \bigg|_4^1 = \frac{2}{7} (1 - 128) = -\frac{254}{7} \).

17. \( \int_1^3 \frac{dt}{t^2} \)  
SOLUTION \( \int_1^3 \frac{dt}{t^2} = \int_1^3 t^{-2} \, dt = -t^{-1} \bigg|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3} \).

18. \( \int_1^4 x^{-4} \, dx \)  
SOLUTION \( \int_1^4 x^{-4} \, dx = -\frac{1}{3} x^{-3} \bigg|_1^4 = -\frac{1}{3} (4)^{-3} + \frac{1}{3} = \frac{21}{64} \).

19. \( \int_{1/2}^1 \frac{8}{x^3} \, dx \)  
SOLUTION \( \int_{1/2}^1 \frac{8}{x^3} \, dx = \int_{1/2}^1 8x^{-3} \, dx = -4x^{-2} \bigg|_{1/2}^1 = -4 + 16 = 12 \).

20. \( \int_{-2}^{-1} \frac{1}{x^3} \, dx \)  
SOLUTION \( \int_{-2}^{-1} \frac{1}{x^3} \, dx = \frac{1}{2} x^{-2} \bigg|_{-2}^{-1} = \frac{1}{2} (-1)^{-2} + \frac{1}{2} (-2)^{-2} = \frac{3}{8} \).

21. \( \int_1^2 (x^2 - x^{-2}) \, dx \)  
SOLUTION \( \int_1^2 (x^2 - x^{-2}) \, dx = \left( \frac{1}{3} x^3 + x^{-1} \right)^2 \bigg|_1^2 = \left( \frac{8}{3} + \frac{1}{2} \right) - \left( \frac{1}{3} + 1 \right) = \frac{11}{6} \).

22. \( \int_1^9 t^{-1/2} \, dt \)  
SOLUTION \( \int_1^9 t^{-1/2} \, dt = 2t^{1/2} \bigg|_1^9 = 2(9)^{1/2} - 2(1)^{1/2} = 4 \).

23. \( \int_1^{27} \frac{t + 1}{\sqrt{t}} \, dt \)  
SOLUTION
\[
\int_1^{27} \frac{t + 1}{\sqrt{t}} \, dt = \int_1^{27} (t^{1/2} + t^{-1/2}) \, dt = \left( \frac{2}{3} t^{3/2} + 2t^{1/2} \right)^{27}_1
= \left( \frac{2}{3} (81\sqrt{3}) + 6\sqrt{3} \right) - \left( \frac{2}{3} + 2 \right) = 60\sqrt{3} - \frac{8}{3}.
\]

24. \( \int_{8/27}^1 \frac{10t^{4/3} - 8t^{1/3}}{t^2} \, dt \)  
SOLUTION
\[
\int_{8/27}^1 \frac{10t^{4/3} - 8t^{1/3}}{t^2} \, dt = \int_{8/27}^1 \frac{10t^{-2/3} - 8t^{-5/3}}{dt} = (30t^{1/3} + 12t^{-2/3}) \bigg|_{8/27}^1 = (30 + 12) - (20 + 27) = -5.
\]
25. \[ \int_{\pi/4}^{3\pi/4} \sin \theta \, d\theta \]

**Solution**

\[ \int_{\pi/4}^{3\pi/4} \sin \theta \, d\theta = -\cos \theta \bigg|_{\pi/4}^{3\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}. \]

26. \[ \int_{2\pi}^{4\pi} \sin x \, dx \]

**Solution**

\[ \int_{2\pi}^{4\pi} \sin x \, dx = -\cos x \bigg|_{2\pi}^{4\pi} = -1 - (-1) = 0. \]

27. \[ \int_{0}^{\pi/2} \cos \left( \frac{1}{3} \theta \right) \, d\theta \]

**Solution**

\[ \int_{0}^{\pi/2} \cos \left( \frac{1}{3} \theta \right) \, d\theta = 3 \sin \left( \frac{1}{3} \theta \right) \bigg|_{0}^{\pi/2} = \frac{3}{2}. \]

28. \[ \int_{\pi/4}^{5\pi/8} \cos 2x \, dx \]

**Solution**

\[ \int_{\pi/4}^{5\pi/8} \cos 2x \, dx = \frac{1}{2} \sin 2x \bigg|_{\pi/4}^{5\pi/8} = \frac{1}{2} \sin \frac{5\pi}{4} - \frac{1}{2} \sin \frac{\pi}{2} = -\frac{\sqrt{2}}{4} - \frac{1}{2} = -\frac{3}{4}. \]

29. \[ \int_{0}^{\pi/6} \sec^2 \left( 3t - \frac{\pi}{6} \right) \, dt \]

**Solution**

\[ \int_{0}^{\pi/6} \sec^2 \left( 3t - \frac{\pi}{6} \right) \, dt = \frac{1}{3} \tan \left( 3t - \frac{\pi}{6} \right) \bigg|_{0}^{\pi/6} = \frac{1}{3} \left( \sqrt{3} + \frac{1}{\sqrt{3}} \right) = \frac{4}{3} \sqrt{3}. \]

30. \[ \int_{0}^{\pi/6} \sec \theta \tan \theta \, d\theta \]

**Solution**

\[ \int_{0}^{\pi/6} \sec \theta \tan \theta \, d\theta = \sec \theta \bigg|_{0}^{\pi/6} = \sec \frac{\pi}{6} - \sec 0 = \frac{2\sqrt{3}}{3} - 1. \]

31. \[ \int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx \]

**Solution**

\[ \int_{\pi/20}^{\pi/10} \csc 5x \cot 5x \, dx = -\frac{1}{5} \csc 5x \bigg|_{\pi/20}^{\pi/10} = -\frac{1}{5} \left( 1 - \sqrt{2} \right) = \frac{1}{5} (\sqrt{2} - 1). \]

32. \[ \int_{\pi/28}^{\pi/14} \csc^2 7y \, dy \]

**Solution**

\[ \int_{\pi/28}^{\pi/14} \csc^2 7y \, dy = \frac{1}{7} \cot 7y \bigg|_{\pi/28}^{\pi/14} = \frac{1}{7} \cot \frac{\pi}{4} - \frac{1}{7} \cot \frac{\pi}{2} = \frac{1}{7}. \]

33. \[ \int_{0}^{1} e^x \, dx \]

**Solution**

\[ \int_{0}^{1} e^x \, dx = e^x \bigg|_{0}^{1} = e - 1. \]

34. \[ \int_{3}^{5} e^{-4x} \, dx \]

**Solution**

\[ \int_{3}^{5} e^{-4x} \, dx = \frac{1}{4} \left( e^{-20} - e^{-12} \right). \]

35. \[ \int_{0}^{3} e^{1-6t} \, dt \]

**Solution**

\[ \int_{0}^{3} e^{1-6t} \, dt = \frac{1}{6} \left( e^{-17} - e^{-17} \right) = \frac{1}{6} (e - e^{-17}). \]

36. \[ \int_{2}^{4} e^{4x-3} \, dx \]
SOLUTION \[
\int_{\frac{3}{2}}^{4} e^{4t-3} \, dt = \left. \frac{1}{4} e^{4t-3} \right|_{\frac{3}{2}}^{4} = \frac{1}{4} e^{4} - \frac{1}{4} e^{-\frac{9}{2}}.
\]

37. \[\int_{2}^{10} \frac{dx}{x}\]

SOLUTION \[\int_{2}^{10} \frac{dx}{x} = \ln |x| \bigg|_{2}^{10} = \ln 10 - \ln 2 = \ln 5.
\]

38. \[\int_{-12}^{-4} \frac{dx}{x}\]

SOLUTION \[\int_{-12}^{-4} \frac{dx}{x} = \ln |x| \bigg|_{-12}^{-4} = \ln |-4| - \ln |-12| = \ln \frac{1}{3} = -\ln 3.
\]

39. \[\int_{0}^{1} \frac{dt}{t+1}\]

SOLUTION \[\int_{0}^{1} \frac{dt}{t+1} = \ln |t+1| \bigg|_{0}^{1} = \ln 2 - \ln 1 = \ln 2.
\]

40. \[\int_{1}^{4} \frac{dt}{5t+4}\]

SOLUTION \[\int_{1}^{4} \frac{dt}{5t+4} = \frac{1}{5} \ln |5t+4| \bigg|_{1}^{4} = \frac{1}{5} \ln 24 - \frac{1}{5} \ln 9 = \frac{1}{5} \ln \frac{24}{9}.
\]

41. \[\int_{-2}^{0} (3x - 9e^{3x}) \, dx\]

SOLUTION \[\int_{-2}^{0} (3x - 9e^{3x}) \, dx = \left. \left( \frac{3}{2} x^2 - 3e^{3x} \right) \right|_{-2}^{0} = (0 - 3) - (6 - 3e^{-6}) = 3e^{-6} - 9.
\]

42. \[\int_{2}^{6} \left( x + \frac{1}{x} \right) \, dx\]

SOLUTION \[\int_{2}^{6} \left( x + \frac{1}{x} \right) \, dx = \left( \frac{1}{2} x^2 + \ln |x| \right) \bigg|_{2}^{6} = (18 + \ln 6) - (2 + \ln 2) = 16 + \ln 3.
\]

In Exercises 43–48, write the integral as a sum of integrals without absolute values and evaluate.

43. \[\int_{-2}^{1} |x| \, dx\]

SOLUTION

\[\int_{-2}^{1} |x| \, dx = \int_{-2}^{0} (-x) \, dx + \int_{0}^{1} x \, dx = \left. \frac{-1}{2} x^2 \right|_{-2}^{0} + \left. \frac{1}{2} x^2 \right|_{0}^{1} = 0 - \left( -\frac{1}{2} (4) \right) + \frac{1}{2} = \frac{5}{2}.
\]

44. \[\int_{0}^{5} |3 - x| \, dx\]

SOLUTION

\[\int_{0}^{5} |3 - x| \, dx = \int_{0}^{3} (3 - x) \, dx + \int_{3}^{5} (x - 3) \, dx = \left. \left( 3x - \frac{1}{2} x^2 \right) \right|_{0}^{3} + \left. \left( \frac{1}{2} x^2 - 3x \right) \right|_{3}^{5}
\]

\[= \left( 9 - \frac{9}{2} \right) - 0 + \left( \frac{25}{2} - 15 \right) - \left( \frac{9}{2} - 9 \right) = \frac{13}{2}.
\]

45. \[\int_{-2}^{3} |x^3| \, dx\]

SOLUTION

\[\int_{-2}^{3} |x^3| \, dx = \int_{-2}^{0} (-x^3) \, dx + \int_{0}^{3} x^3 \, dx = \left. \frac{-1}{4} x^4 \right|_{-2}^{0} + \left. \frac{1}{4} x^4 \right|_{0}^{3}
\]

\[= 0 + \frac{1}{4} (-2)^4 + \frac{1}{4} 3^4 - 0 = \frac{97}{4}.
\]
46. \( \int_0^3 |x^2 - 1| \, dx \)

**SOLUTION**

\[
\int_0^3 |x^2 - 1| \, dx = \int_0^1 (1 - x^2) \, dx + \int_1^3 (x^2 - 1) \, dx = \left( x - \frac{1}{3}x^3 \right) \bigg|_0^1 + \left( \frac{1}{3}x^3 - x \right) \bigg|_1^3 \\
= \left( 1 - \frac{1}{3} \right) - 0 + (9 - 3) - \left( \frac{1}{3} - 1 \right) = \frac{22}{3}.
\]

47. \( \int_0^\pi |\cos x| \, dx \)

**SOLUTION**

\[
\int_0^\pi |\cos x| \, dx = \int_0^{\pi/2} \cos x \, dx + \int_0^{\pi/2} (-\cos x) \, dx = \sin x \bigg|_0^{\pi/2} - \sin x \bigg|_0^{\pi/2} = 1 - 0 - (-1 - 0) = 2.
\]

48. \( \int_0^5 |x^2 - 4x + 3| \, dx \)

**SOLUTION**

\[
\int_0^5 |x^2 - 4x + 3| \, dx = \int_0^5 |(x - 3)(x - 1)| \, dx \\
= \int_0^1 (x^2 - 4x + 3) \, dx + \int_1^3 -(x^2 - 4x + 3) \, dx + \int_3^5 (x^2 - 4x + 3) \, dx \\
= \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_0^1 - \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_1^3 + \left( \frac{1}{3}x^3 - 2x^2 + 3x \right) \bigg|_3^5 \\
= \left( \frac{1}{3} - 2 + 3 \right) - 0 - (9 - 18 + 9) + \left( \frac{1}{3} - 2 + 3 \right) + \left( \frac{125}{3} - 50 + 15 \right) - (9 - 18 + 9) \\
= \frac{28}{3}.
\]

In Exercises 49–54, evaluate the integral in terms of the constants.

49. \( \int_1^b x^3 \, dx \)

**SOLUTION**

\[
\int_1^b x^3 \, dx = \frac{1}{4}x^4 \bigg|_1^b = \frac{1}{4}b^4 - \frac{1}{4}(1)^4 = \frac{1}{4} \left( b^4 - 1 \right)
\]

for any number \( b \).

50. \( \int_b^a x^4 \, dx \)

**SOLUTION**

\[
\int_b^a x^4 \, dx = \frac{1}{5}x^5 \bigg|_b^a = \frac{1}{5}a^5 - \frac{1}{5}b^5
\]

for any numbers \( a, b \).

51. \( \int_1^b x^5 \, dx \)

**SOLUTION**

\[
\int_1^b x^5 \, dx = \frac{1}{6}x^6 \bigg|_1^b = \frac{1}{6}b^6 - \frac{1}{6}(1)^6 = \frac{1}{6}(b^6 - 1)
\]

for any number \( b \).

52. \( \int_{-\infty}^x (t^3 + t) \, dt \)

**SOLUTION**

\[
\int_{-\infty}^x (t^3 + t) \, dt = \left( \frac{1}{4}t^4 + \frac{1}{2}t^2 \right) \bigg|_{-\infty}^x = \left( \frac{1}{4}x^4 + \frac{1}{2}x^2 \right) - \left( \frac{1}{4}(-\infty)^4 + \frac{1}{2}(-\infty)^2 \right) = 0.
\]

53. \( \int_a^b \frac{dx}{x} \)

**SOLUTION**

\[
\int_a^b \frac{dx}{x} = \ln |x| \bigg|_a^b = \ln |5a| - \ln |a| = \ln 5.
\]
54. \( \int_{b}^{b^2} \frac{dx}{x} \)

**SOLUTION**
\[
\int_{b}^{b^2} \frac{dx}{x} = \ln |x| \bigg|_{b}^{b^2} = \ln |b^2| - \ln |b| = \ln |b|.
\]

55. Calculate \( \int_{-2}^{3} f(x) \, dx \), where

\[
f(x) = \begin{cases} 
12 - x^2 & \text{for } x \leq 2 \\
3x^3 & \text{for } x > 2
\end{cases}
\]

**SOLUTION**
\[
\int_{-2}^{3} f(x) \, dx = \int_{-2}^{2} f(x) \, dx + \int_{2}^{3} f(x) \, dx = \int_{-2}^{2} (12 - x^2) \, dx + \int_{2}^{3} 3x^3 \, dx \\
= \left[ 12x - \frac{1}{3}x^3 \right]_{-2}^{2} + \left[ x^4 \right]_{2}^{3} \\
= \left( 12(2) - \frac{1}{3}(2)^3 \right) - \left( 12(-2) - \frac{1}{3}(-2)^3 \right) + \frac{1}{4}(3^4) - \frac{1}{4}(2^4) \\
= \frac{128}{3} + \frac{65}{4} = \frac{707}{12}.
\]

56. Calculate \( \int_{0}^{2\pi} f(x) \, dx \), where

\[
f(x) = \begin{cases} 
\cos x & \text{for } x \leq \pi \\
\cos x - \sin 2x & \text{for } x > \pi
\end{cases}
\]

**SOLUTION**
\[
\int_{0}^{2\pi} f(x) \, dx = \int_{0}^{\pi} f(x) \, dx + \int_{\pi}^{2\pi} f(x) \, dx = \int_{0}^{\pi} \cos x \, dx + \int_{\pi}^{2\pi} (\cos x - \sin 2x) \, dx \\
= \sin x \bigg|_{0}^{\pi} + \left( \sin x + \frac{1}{2} \cos 2x \right) \bigg|_{\pi}^{2\pi} \\
= (0 - 0) + \left( 0 + \frac{1}{2} \right) - \left( 0 + \frac{1}{2} \right) = 0.
\]

57. Use FTC I to show that \( \int_{-1}^{1} x^n \, dx = 0 \) if \( n \) is an odd whole number. Explain graphically.

**SOLUTION** We have
\[
\int_{-1}^{1} x^n \, dx = \frac{x^{n+1}}{n+1} \bigg|_{-1}^{1} = \frac{(1)^{n+1} - (-1)^{n+1}}{n+1}.
\]

Because \( n \) is odd, \( n + 1 \) is even, which means that \((-1)^{n+1} = (1)^{n+1} = 1\). Hence
\[
\frac{(1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0.
\]

Graphically speaking, for an odd function such as \( x^3 \) shown here, the positively signed area from \( x = 0 \) to \( x = 1 \) cancels the negatively signed area from \( x = -1 \) to \( x = 0 \).

![Graphical representation](image.png)
58. **CRA** Plot the function \( f(x) = 3x - x \). Find the positive root of \( f(x) \) to three places and use it to find the area under the graph of \( f(x) \) in the first quadrant.

**SOLUTION** The graph of \( f(x) = 3x - x \) is shown below at the left. In the figure below at the right, we zoom in on the positive root of \( f(x) \) and find that, to three decimal places, this root is approximately \( x = 0.760 \). The area under the graph of \( f(x) \) in the first quadrant is then

\[
\int_0^0.760 (3x - x) \, dx = \left( -\frac{1}{3} \cos 3x - \frac{1}{2} x^2 \right) \bigg|_0^{0.760} = -\frac{1}{3} \cos (2.28) - \frac{1}{2} (0.760)^2 + \frac{1}{3} \approx 0.262
\]

59. Calculate \( F(4) \) given that \( F(1) = 3 \) and \( F'(x) = x^2 \). Hint: Express \( F(4) - F(1) \) as a definite integral.

**SOLUTION** By FTC I,

\[
F(4) - F(1) = \int_1^4 x^2 \, dx = \frac{4^3 - 1^3}{3} = 21
\]

Therefore \( F(4) = F(1) + 21 = 3 + 21 = 24 \).

60. Calculate \( G(16) \), where \( dG/dt = t^{-1/2} \) and \( G(9) = -5 \).

**SOLUTION** By FTC I,

\[
G(16) - G(9) = \int_9^{16} t^{-1/2} \, dt = 2(16^{1/2}) - 2(9^{1/2}) = 2
\]

Therefore \( G(16) = -5 + 2 = -3 \).

61. Does \( \int_0^1 x^n \, dx \) get larger or smaller as \( n \) increases? Explain graphically.

**SOLUTION** Let \( n \geq 0 \) and consider \( \int_0^1 x^n \, dx \). (Note: for \( n < 0 \) the integrand \( x^n \to \infty \) as \( x \to 0+ \), so we exclude this possibility.) Now

\[
\int_0^1 x^n \, dx = \left( \frac{1}{n+1} x^{n+1} \right) \bigg|_0^1 = \frac{1}{n+1} (1^{n+1}) - \frac{1}{n+1} (0^{n+1}) = \frac{1}{n+1}
\]

which decreases as \( n \) increases. Recall that \( \int_0^1 x^n \, dx \) represents the area between the positive curve \( f(x) = x^n \) and the \( x \)-axis over the interval \([0, 1]\). Accordingly, this area gets smaller as \( n \) gets larger. This is readily evident in the following graph, which shows curves for several values of \( n \).

62. Show that the area of the shaded parabolic arch in Figure 1 is equal to four-thirds the area of the triangle shown.

**FIGURE 1** Graph of \( y = (x - a)(b - x) \).
SOLUTION We first calculate the area of the parabolic arch:

\[
\int_a^b (x-a)(b-x) \, dx = - \int_a^b (x-a)(x-b) \, dx = - \int_a^b (x^2 - ax - bx + ab) \, dx
\]

\[
= - \left( \frac{1}{2} x^2 - \frac{a}{2} x^2 - \frac{b}{2} x^2 + abx \right) \bigg|_a^b
\]

\[
= - \frac{1}{6} \left( 2x^3 - 3ax^2 - 3bx^2 + 6abx \right) \bigg|_a^b
\]

\[
= - \frac{1}{6} \left( (2b^3 - 3ab^2 - 3b^3 + 6ab^2) - (2a^3 - 3a^2b^2 + 6a^2b) \right)
\]

\[
= - \frac{1}{6} \left( (b^3 - 3ab^2) - (a^3 - 3a^2b + 6a^2b) \right)
\]

\[
= - \frac{1}{6} \left( a^3 + 3ab^2 - 3a^2b - b^3 \right) = \frac{1}{6} (b-a)^3.
\]

The indicated triangle has a base of length \( b - a \) and a height of

\[
\left( \frac{a+b}{2} - a \right) \left( b - \frac{a+b}{2} \right) = \left( \frac{b-a}{2} \right)^2.
\]

Thus, the area of the triangle is

\[
\frac{1}{2} (b-a) \left( \frac{b-a}{2} \right)^2 = \frac{1}{8} (b-a)^3.
\]

Finally, we note that

\[
\frac{1}{6} (b-a)^3 = \frac{4}{3} \cdot \frac{1}{8} (b-a)^3,
\]

as required.

Further Insights and Challenges

63. Prove a famous result of Archimedes (generalizing Exercise 62): For \( r < s \), the area of the shaded region in Figure 2 is equal to four-thirds the area of triangle \( \triangle ACE \), where \( C \) is the point on the parabola at which the tangent line is parallel to secant line \( AE \).

(a) Show that \( C \) has \( x \)-coordinate \( (r + s)/2 \).

(b) Show that \( ABDE \) has area \( (s - r)^3/4 \) by viewing it as a parallelogram of height \( s - r \) and base of length \( CF \).

(c) Show that \( \triangle ACE \) has area \( (s - r)^3/8 \) by observing that it has the same base and height as the parallelogram.

(d) Compute the shaded area as the area under the graph minus the area of a trapezoid, and prove Archimedes’ result.

\[ \text{FIGURE 2 Graph of } f(x) = (x-a)(b-x). \]

SOLUTION

(a) The slope of the secant line \( \overline{AE} \) is

\[
\frac{f(s) - f(r)}{s - r} = \frac{(s-a)(b-s) - (r-a)(b-r)}{s - r} = a + b - (r + s)
\]

and the slope of the tangent line along the parabola is

\[
f'(x) = a + b - 2x.
\]

If \( C \) is the point on the parabola at which the tangent line is parallel to the secant line \( \overline{AE} \), then its \( x \)-coordinate must satisfy

\[
a + b - 2x = a + b - (r + s) \quad \text{or} \quad x = \frac{r + s}{2}.
\]
(b) Parallelogram $ABDE$ has height $s - r$ and base of length $\overline{CF}$. Since the equation of the secant line $\overline{AE}$ is 

$$y = [a + b - (r + s)](x - r) + (r - a)(b - r),$$

the length of the segment $\overline{CF}$ is

$$\left( \frac{r + s}{2} - a \right) \left( b - \frac{r + s}{2} \right) - [a + b - (r + s)] \left( \frac{r + s}{2} - r \right) - (r - a)(b - r) = \frac{(s - r)^2}{4}.$$ 

Thus, the area of $ABDE$ is $\frac{(s - r)^3}{3}$.

(c) Triangle $ACE$ is comprised of $\Delta ACF$ and $\Delta CEF$. Each of these smaller triangles has height $\frac{s - r}{2}$ and base of length $\frac{(s - r)^2}{4}$. Thus, the area of $\Delta ACE$ is

$$\frac{1}{2}(s - r) \cdot \frac{(s - r)^2}{4} + \frac{1}{2}(s - r) \cdot \frac{(s - r)^2}{4} = \frac{(s - r)^3}{8}.$$ 

(d) The area under the graph of the parabola between $x = r$ and $x = s$ is

$$\int_r^s (x - a)(b - x) \, dx = \left[ -abx + \frac{1}{2}(a + b)x^2 - \frac{1}{3}x^3 \right]_r^s$$

$$= -abs + \frac{1}{2}(a + b)s^2 - \frac{1}{3}s^3 + abr - \frac{1}{2}(a + b)r^2 + \frac{1}{3}r^3$$

$$= ab(r - s) + \frac{1}{2}(a + b)(s - r)(s + r) + \frac{1}{3}(r - s)(r^2 + rs + s^2),$$

while the area of the trapezoid under the shaded region is

$$\frac{1}{2}(s - r) \cdot [(s - a)(b - s) + (r - a)(b - r)]$$

$$= \frac{1}{2}(s - r) \left[ -2ab + (a + b)(r + s) - r^2 - s^2 \right]$$

$$= ab(r - s) + \frac{1}{2}(a + b)(s - r)(s + r) + \frac{1}{2}(r - s)(r^2 + rs + s^2).$$

Thus, the area of the shaded region is

$$(r - s) \left( \frac{1}{3}r^2 + \frac{1}{3}rs + \frac{1}{3}s^2 - \frac{1}{2}r^2 - \frac{1}{2}s^2 \right) = (s - r) \left( \frac{1}{6}r^2 - \frac{1}{3}rs + \frac{1}{6}s^2 \right) = \frac{1}{6}(s - r)^3,$$

which is four-thirds the area of the triangle $ACE$.

64. (a) Apply the Comparison Theorem (Theorem 5 in Section 5.2) to the inequality $\sin x \leq x$ (valid for $x \geq 0$) to prove that

$$1 - \frac{x^2}{2} \leq \cos x \leq 1$$

(b) Apply it again to prove that

$$x - \frac{x^3}{6} \leq \sin x \leq x \quad \text{(for } x \geq 0)$$

(c) Verify these inequalities for $x = 0.3$.

SOLUTION

(a) We have $\int_0^x \sin t \, dt = -\cos t \bigg|_0^x = -\cos x + 1$ and $\int_0^x t \, dt = \frac{1}{2}t^2 \bigg|_0^x = \frac{1}{2}x^2$. Hence

$$-\cos x + 1 \leq \frac{x^2}{2}.$$ 

Solving, this gives $\cos x \geq 1 - \frac{x^2}{2}$, $\cos x \leq 1$ follows automatically.

(b) The previous part gives $1 - \frac{x^2}{2} \leq \cos t \leq 1$, for $t > 0$. Theorem 5 gives us, after integrating over the interval $[0, x]$,

$$x - \frac{x^3}{6} \leq \sin x \leq x.$$

(c) Substituting $x = 0.3$ into the inequalities obtained in (a) and (b) yields

$$0.955 \leq 0.955336489 \leq 1 \quad \text{and} \quad 0.2955 \leq 0.2955202069 \leq 0.3,$$

respectively.
65. Use the method of Exercise 64 to prove that

\[
1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24},
\]

\[
x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120} \quad (\text{for } x \geq 0)
\]

Verify these inequalities for \( x = 0.1 \). Why have we specified \( x \geq 0 \) for \( \sin x \) but not for \( \cos x \)?

**SOLUTION** By Exercise 64, \( t - \frac{t^3}{6} \leq \sin t \leq t \) for \( t > 0 \). Integrating this inequality over \( [0, x] \), and then solving for \( \cos x \), yields:

\[
\frac{1}{2} x^2 - \frac{1}{24} x^4 \leq 1 - \cos x \leq \frac{1}{2} x^2
\]

\[
1 - \frac{1}{2} x^2 \leq \cos x \leq 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4.
\]

These inequalities apply for \( x \geq 0 \). Since \( \cos x, 1 - \frac{x^2}{2} \), and \( 1 - \frac{x^2}{2} + \frac{x^4}{24} \) are all even functions, they also apply for \( x \leq 0 \). Having established that

\[
1 - \frac{t^2}{2} \leq \cos t \leq 1 - \frac{t^2}{2} + \frac{t^4}{24},
\]

for all \( t \geq 0 \), we integrate over the interval \( [0, x] \), to obtain:

\[
x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.
\]

The functions \( \sin x, x - \frac{1}{6} x^3 \) and \( x - \frac{1}{6} x^3 + \frac{1}{120} x^5 \) are all odd functions, so the inequalities are reversed for \( x < 0 \). Evaluating these inequalities at \( x = 0.1 \) yields

\[
0.995000000 \leq 0.995004165 \leq 0.995004167
\]

\[
0.999833333 \leq 0.9998334166 \leq 0.9998334167,
\]

both of which are true.

66. Calculate the next pair of inequalities for \( \sin x \) and \( \cos x \) by integrating the results of Exercise 65. Can you guess the general pattern?

**SOLUTION** Integrating

\[
t - \frac{t^3}{6} \leq \sin t \leq t - \frac{t^3}{6} + \frac{t^5}{120} \quad (\text{for } t \geq 0)
\]

over the interval \( [0, x] \) yields

\[
\frac{x^2}{2} - \frac{x^4}{24} \leq 1 - \cos x \leq \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720}.
\]

Solving for \( \cos x \) yields

\[
1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.
\]

Replacing each \( x \) by \( t \) and integrating over the interval \( [0, t] \) produces

\[
x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.
\]

To see the pattern, it is best to compare consecutive inequalities for \( \sin x \) and those for \( \cos x \):

\[
0 \leq \sin x \leq x
\]

\[
x - \frac{x^3}{6} \leq \sin x \leq x
\]

\[
x - \frac{x^3}{6} \leq \sin x \leq x - \frac{x^3}{6} + \frac{x^5}{120}.
\]
Each iteration adds an additional term. Looking at the highest order terms, we get the following pattern:

\[
\begin{align*}
0 \\
x \\
\frac{x^3}{6} = -\frac{x^3}{3!} \\
\frac{x^5}{5!}
\end{align*}
\]

We guess that the leading term of the polynomials are of the form

\[(-1)^n \frac{x^{2n+1}}{(2n+1)!}\]

Similarly, for \(\cos x\), the leading terms of the polynomials in the inequality are of the form

\[(-1)^n \frac{x^{2n}}{(2n)!}\]

67. Use FTC I to prove that if \(|f'(x)| \leq K\) for \(x \in [a, b]\), then \(|f(x) - f(a)| \leq K|x - a|\) for \(x \in [a, b]\).

**SOLUTION** Let \(a > b\) be real numbers, and let \(f(x)\) be such that \(|f'(x)| \leq K\) for \(x \in [a, b]\). By FTC,

\[
\int_a^x f'(t)\,dt = f(x) - f(a).
\]

Since \(f'(x) \geq -K\) for all \(x \in [a, b]\), we get:

\[
f(x) - f(a) = \int_a^x f'(t)\,dt \geq -K(x - a).
\]

Since \(f'(x) \leq K\) for all \(x \in [a, b]\), we get:

\[
f(x) - f(a) = \int_a^x f'(t)\,dt \leq K(x - a).
\]

Combining these two inequalities yields

\[-K(x - a) \leq f(x) - f(a) \leq K(x - a),\]

so that, by definition,

\[|f(x) - f(a)| \leq K|x - a|.
\]

68. (a) Use Exercise 67 to prove that \(|\sin a - \sin b| \leq |a - b|\) for all \(a, b\).
(b) Let \(f(x) = \sin(x + a) - \sin x\). Use part (a) to show that the graph of \(f(x)\) lies between the horizontal lines \(y = \pm a\).
(c) Plot \(f(x)\) and the lines \(y = \pm a\) to verify (b) for \(a = 0.5\) and \(a = 0.2\).

**SOLUTION**

(a) Let \(f(x) = \sin x\), so that \(f'(x) = \cos x\), and

\[|f'(x)| \leq 1\]

for all \(x\). From Exercise 67, we get:

\[|\sin a - \sin b| \leq |a - b|.
\]

(b) Let \(f(x) = \sin(x + a) - \sin x\). Applying (a), we get the inequality:

\[|f(x)| = |\sin(x + a) - \sin x| \leq |x + a - x| = |a|.
\]

This is equivalent, by definition, to the two inequalities:

\[-a \leq \sin(x + a) - \sin x \leq a.
\]

(c) The plots of \(y = \sin(x + 0.5) - \sin x\) and of \(y = \sin(x + 0.2) - \sin x\) are shown below. The inequality is satisfied in both plots.