4.5 L'Hôpital’s Rule

Preliminary Questions

1. What is wrong with applying L'Hôpital’s Rule to \( \lim_{x \to 0} \frac{x^2 - 2x}{3x - 2} \)?

**SOLUTION** As \( x \to 0 \),

\[
\lim_{x \to 0} \frac{x^2 - 2x}{3x - 2} = \frac{0 - 0}{0 - 2} = -\frac{1}{2},
\]

is not of the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule cannot be used.

2. Does L'Hôpital’s Rule apply to \( \lim_{x \to a} f(x)g(x) \) if \( f(x) \) and \( g(x) \) both approach \( \infty \) as \( x \to a \)?

**SOLUTION** No. L'Hôpital’s Rule only applies to limits of the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

Exercises

In Exercises 1–10, use L'Hôpital’s Rule to evaluate the limit, or state that L'Hôpital’s Rule does not apply.

1. \( \lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 4} \)

**SOLUTION** Because the quotient is not indeterminate at \( x = 3 \),

\[
\lim_{x \to 3} \frac{2x^2 - 5x - 3}{x - 4} = \frac{0}{1} = 0.
\]

L'Hôpital’s Rule does not apply.

2. \( \lim_{x \to 5} \frac{x^2 - 25}{5 - 4x - x^2} \)

**SOLUTION** The functions \( x^2 - 25 \) and \( 5 - 4x - x^2 \) are differentiable, but the quotient is indeterminate at \( x = -5 \),

\[
\lim_{x \to -5} \frac{x^2 - 25}{5 - 4x - x^2} = \lim_{x \to -5} \frac{2x}{-4 - 2x} = \frac{-10}{-10} = 1.
\]

so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to -5} \frac{x^2 - 25}{5 - 4x - x^2} = \lim_{x \to -5} \frac{2x}{-4 - 2x} = \frac{-10}{-10} = 1.
\]

3. \( \lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16} \)

**SOLUTION** Because the quotient is not indeterminate at \( x = 4 \),

\[
\lim_{x \to 4} \frac{x^3 - 64}{x^2 + 16} = \frac{64 - 64}{16 + 16} = 0.
\]

L'Hôpital’s Rule does not apply.

4. \( \lim_{x \to -1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1} \)

**SOLUTION** The functions \( x^4 + 2x + 1 \) and \( x^5 - 2x - 1 \) are differentiable, but the quotient is indeterminate at \( x = -1 \),

\[
\lim_{x \to -1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1} = \lim_{x \to -1} \frac{4x^3 + 2}{5x^4 - 2} = \frac{4 + 2}{5 - 2} = \frac{2}{3}.
\]

so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to -1} \frac{x^4 + 2x + 1}{x^5 - 2x - 1} = \lim_{x \to -1} \frac{4x^3 + 2}{5x^4 - 2} = \frac{4 + 2}{5 - 2} = \frac{2}{3}.
\]
5. \( \lim_{x \to 9} \frac{x^{1/2} + x - 6}{x^{3/2} - 27} \)

**Solution**  Because the quotient is not indeterminate at \( x = 9 \),

\[
\frac{x^{1/2} + x - 6}{x^{3/2} - 27} \bigg|_{x=9} = \frac{3 + 9 - 6}{27 - 27} = \frac{6}{0}
\]

L'Hôpital’s Rule does not apply.

6. \( \lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x^3 - 7x - 6} \)

**Solution**  The functions \( \sqrt{x + 1} - 2 \) and \( x^3 - 7x - 6 \) are differentiable, but the quotient is indeterminate at \( x = 3 \),

\[
\frac{\sqrt{x + 1} - 2}{x^3 - 7x - 6} \bigg|_{x=3} = \frac{2 - 2}{27 - 21 - 6} = \frac{0}{0},
\]

so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x^3 - 7x - 6} = \frac{1}{2 \sqrt{x + 1}} \cdot \frac{3x^2 - 7}{3x^2 - 7} = \frac{1}{3} = \frac{1}{80}.
\]

7. \( \lim_{x \to 0} \frac{\sin 4x}{x^2 + 3x + 1} \)

**Solution**  Because the quotient is not indeterminate at \( x = 0 \),

\[
\frac{\sin 4x}{x^2 + 3x + 1} \bigg|_{x=0} = \frac{0}{0 + 1} = 0.
\]

L'Hôpital’s Rule does not apply.

8. \( \lim_{x \to 0} \frac{x^3}{\sin x - x} \)

**Solution**  The functions \( x^3 \) and \( \sin x - x \) are differentiable, but the quotient is indeterminate at \( x = 0 \),

\[
\frac{x^3}{\sin x - x} \bigg|_{x=0} = \frac{0}{0 - 0} = \frac{0}{0},
\]

so L'Hôpital’s Rule applies. Here, we use L'Hôpital's Rule three times to find

\[
\lim_{x \to 0} \frac{x^3}{\sin x - x} = \lim_{x \to 0} \frac{3x^2}{\cos x - 1} = \lim_{x \to 0} \frac{6x}{x - \sin x} = \lim_{x \to 0} \frac{6}{x - \cos x} = -6.
\]

9. \( \lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} \)

**Solution**  The functions \( \cos 2x - 1 \) and \( \sin 5x \) are differentiable, but the quotient is indeterminate at \( x = 0 \),

\[
\frac{\cos 2x - 1}{\sin 5x} \bigg|_{x=0} = \frac{1 - 1}{0} = \frac{0}{0},
\]

so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to 0} \frac{\cos 2x - 1}{\sin 5x} = \lim_{x \to 0} \frac{-2 \sin 2x}{5 \cos 5x} = \frac{0}{0} = 0.
\]

10. \( \lim_{x \to 0} \frac{\cos x - \sin^2 x}{\sin x} \)

**Solution**  Because the quotient is not indeterminate at \( x = 0 \),

\[
\frac{\cos x - \sin^2 x}{\sin x} \bigg|_{x=0} = \frac{1 - 0}{0} = \frac{1}{0},
\]

L'Hôpital’s Rule does not apply.
In Exercises 11–16, show that L'Hôpital’s Rule is applicable to the limit as \( x \to \pm \infty \) and evaluate.

11. \( \lim_{x \to \infty} \frac{9x + 4}{3 - 2x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{9x + 4}{3 - 2x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to \infty} \frac{9x + 4}{3 - 2x} = \lim_{x \to \infty} \frac{9}{-2} = \frac{9}{-2}.
\]

12. \( \lim_{x \to \infty} x \sin \frac{1}{x} \)

**SOLUTION** As \( x \to \infty \), \( x \sin \frac{1}{x} \) is of the form \( \infty \cdot 0 \), so L'Hôpital’s Rule does not immediately apply. If we rewrite \( x \sin \frac{1}{x} \) as \( \frac{\sin(1/x)}{1/x} \), the rewritten expression is of the form \( \frac{0}{0} \) as \( x \to \infty \), so L'Hôpital’s Rule now applies. We find

\[
\lim_{x \to \infty} x \cdot \sin \left( \frac{1}{x} \right) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \to \infty} \frac{\cos(1/x)(-1/x^2)}{-1/x^2} = \lim_{x \to \infty} \cos(1/x) = \cos 0 = 1.
\]

13. \( \lim_{x \to \infty} \ln \frac{x}{\sqrt{x}} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{\ln x}{x^{1/2}} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to \infty} \ln \frac{x}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x \to \infty} \frac{1}{x^{1/2}} = 0.
\]

14. \( \lim_{x \to \infty} \frac{x}{e^x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{x}{e^x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule applies. We find

\[
\lim_{x \to \infty} \frac{x}{e^x} = \lim_{x \to \infty} \frac{1}{e^x} = 0.
\]

15. \( \lim_{x \to \infty} \frac{\ln(x^4 + 1)}{x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{\ln(x^4 + 1)}{x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule applies. Here, we use L'Hôpital’s Rule twice to find

\[
\lim_{x \to \infty} \frac{\ln(x^4 + 1)}{x} = \lim_{x \to \infty} \frac{4x^3}{x^4 + 1} = \lim_{x \to \infty} \frac{12x^2}{4x^3} = \lim_{x \to \infty} \frac{3}{x} = 0.
\]

16. \( \lim_{x \to \infty} \frac{x^2}{e^x} \)

**SOLUTION** As \( x \to \infty \), the quotient \( \frac{x^2}{e^x} \) is of the form \( \frac{\infty}{\infty} \), so L'Hôpital’s Rule applies. Here, we use L'Hôpital’s Rule twice to find

\[
\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0.
\]

In Exercises 17–54, evaluate the limit.

17. \( \lim_{x \to 1} \frac{\sqrt{8 + x - 3x^{1/3}}}{x^2 - 3x + 2} \)

**SOLUTION**

\[
\lim_{x \to 1} \frac{\sqrt{8 + x - 3x^{1/3}}}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{\frac{1}{2}(8 + x)^{-1/2} - x^{-2/3}}{2x - 3} = \frac{\frac{1}{2} - 1}{-1} = \frac{5}{6}.
\]

18. \( \lim_{x \to 4} \left[ \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right] \)

**SOLUTION**

\[
\lim_{x \to 4} \left[ \frac{1}{\sqrt{x} - 2} - \frac{4}{x - 4} \right] = \lim_{x \to 4} \left[ \frac{\sqrt{x} + 2}{x - 4} - \frac{4}{x - 4} \right] = \lim_{x \to 4} \frac{\sqrt{x} + 2}{1} = 1.
\]
19. \[ \lim_{x \to \infty} \frac{3x - 2}{1 - 5x} \]

**SOLUTION** \[ \lim_{x \to \infty} \frac{3x - 2}{1 - 5x} = \lim_{x \to \infty} \frac{3}{-5} = \frac{-3}{5} \]

20. \[ \lim_{x \to \infty} \frac{x^{2/3} + 3x}{x^{5/3} - x} \]

**SOLUTION** \[ \lim_{x \to \infty} \frac{x^{2/3} + 3x}{x^{5/3} - x} = \lim_{x \to \infty} \frac{\frac{1}{x^{1/3}} + \frac{3}{x^{2/3}}}{1 - \frac{1}{x^{2/3}}} = 0 + 0 = 0. \]

21. \[ \lim_{x \to \infty} \frac{7x^2 + 4x}{9 - 3x^2} \]

**SOLUTION** \[ \lim_{x \to \infty} \frac{7x^2 + 4x}{9 - 3x^2} = \lim_{x \to \infty} \frac{14x + 4}{-6x} = \lim_{x \to \infty} \frac{14}{-6} = \frac{-7}{3} \]

22. \[ \lim_{x \to \infty} \frac{3x^3 + 4x^2}{4x^3 - 7} \]

**SOLUTION** \[ \lim_{x \to \infty} \frac{3x^3 + 4x^2}{4x^3 - 7} = \lim_{x \to \infty} \frac{9x^2 + 8x}{12x^2} = \lim_{x \to \infty} \frac{18x + 8}{24x} = \frac{18}{24} = \frac{3}{4} \]

23. \[ \lim_{x \to 1} \frac{(1 + 3x)^{1/2} - 2}{(1 + 7x)^{1/3} - 2} \]

**SOLUTION** Apply L'Hôpital's Rule once:

\[ \lim_{x \to 1} \frac{(1 + 3x)^{1/2} - 2}{(1 + 7x)^{1/3} - 2} = \lim_{x \to 1} \frac{\frac{3}{2}(1 + 3x)^{-1/2}}{\frac{7}{3}(1 + 7x)^{-2/3}} \]

\[ = \frac{(\frac{3}{2})(1)}{(\frac{7}{3})(1)} = \frac{9}{7} \]

24. \[ \lim_{x \to 8} \frac{x^{5/3} - 2x - 16}{x^{1/3} - 2} \]

**SOLUTION** \[ \lim_{x \to 8} \frac{x^{5/3} - 2x - 16}{x^{1/3} - 2} = \lim_{x \to 8} \frac{\frac{5}{3}x^{2/3} - 2}{\frac{1}{3}x^{-2/3}} = \frac{20}{2} = 10 \]

25. \[ \lim_{x \to 0} \frac{\sin 2x}{\sin 7x} \]

**SOLUTION** \[ \lim_{x \to 0} \frac{\sin 2x}{\sin 7x} = \lim_{x \to 0} \frac{2 \cos 2x}{7 \cos 7x} = \frac{2}{7} \]

26. \[ \lim_{x \to \pi/2} \frac{\tan 4x}{\tan 5x} \]

**SOLUTION** \[ \lim_{x \to \pi/2} \frac{\tan 4x}{\tan 5x} = \lim_{x \to \pi/2} \frac{4 \sec^2 4x}{5 \sec^2 5x} = \lim_{x \to \pi/2} \frac{4 \cos^2 5x}{5 \cos^2 4x} \]

\[= \lim_{x \to \pi/2} \frac{4 \cos 5x}{5 \cos 4x} = \lim_{x \to \pi/2} \frac{-10 \sin 5x \cos 5x}{-8 \sin 4x \cos 4x} = \lim_{x \to \pi/2} \frac{\sin 10x}{8 \cos 8x} = \frac{5}{4} \]

27. \[ \lim_{x \to 0} \frac{\tan x}{x} \]

**SOLUTION** \[ \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sec^2 x}{1} = 1 \]

28. \[ \lim_{x \to 0} \left( \cot x - \frac{1}{x} \right) \]

**SOLUTION** \[ \lim_{x \to 0} \left( \cot x - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x \cos x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{-x \sin x + \cos x - \cos x}{x \cos x + \sin x} = \lim_{x \to 0} \frac{-x \sin x}{x \cos x + \sin x} \]

\[= \lim_{x \to 0} \frac{-x \cos x - x}{-x \sin x + \cos x + \cos x} = \frac{0}{2} = 0. \]
29. \[ \lim_{x \to 0} \frac{\sin x - x \cos x}{x - \sin x} \]

**SOLUTION**

\[
\lim_{x \to 0} \frac{\sin x - x \cos x}{x - \sin x} = \lim_{x \to 0} \frac{x \sin x}{1 - \cos x} = \lim_{x \to 0} \frac{\sin x + x \cos x}{\sin x} = \lim_{x \to 0} \frac{\cos x + \cos x - x \sin x}{\cos x} = 2.
\]

30. \[ \lim_{x \to \pi/2} (x - \frac{\pi}{2}) \tan x \]

**SOLUTION**

\[
\lim_{x \to \pi/2} (x - \frac{\pi}{2}) \tan x = \lim_{x \to \pi/2} \frac{x - \pi/2}{\tan x} = \lim_{x \to \pi/2} \frac{\pi/2 - \pi/2}{\cot x} = \lim_{x \to \pi/2} \frac{1}{\csc^2 x} = \lim_{x \to \pi/2} -\sin^2 x = -1.
\]

31. \[ \lim_{x \to 0} \frac{\cos(x + \frac{\pi}{4})}{\sin x} \]

**SOLUTION**

\[
\lim_{x \to 0} \frac{\cos(x + \frac{\pi}{4})}{\sin x} = \lim_{x \to 0} \frac{-\sin(x + \frac{\pi}{4})}{\cos x} = -1.
\]

32. \[ \lim_{x \to 0} \frac{x^2}{1 - \cos x} \]

**SOLUTION**

\[
\lim_{x \to 0} \frac{x^2}{1 - \cos x} = \lim_{x \to 0} \frac{2x}{sin x} = \lim_{x \to 0} \frac{2}{\cos x} = 2.
\]

33. \[ \lim_{x \to \pi/2} \cos x \] 

**SOLUTION**

\[
\lim_{x \to \pi/2} \cos x = \lim_{x \to \pi/2} -\sin x \cdot 2\cos(2x) = \frac{1}{2}.
\]

34. \[ \lim_{x \to 0} \left( \frac{1}{x^2} - \csc^2 x \right) \]

**SOLUTION**

\[
\lim_{x \to 0} \left( \frac{1}{x^2} - \csc^2 x \right) = \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{2 \sin x \cos x - 2x}{2x^2 \sin x \cos x + 2x \sin^2 x} = \lim_{x \to 0} \frac{\sin 2x - 2x}{2x^2 \sin 2x + 2x \sin^2 x} = \lim_{x \to 0} \frac{\cos 2x - 1}{2x^2 \cos 2x + 2x \sin 2x + 4x \sin x \cos x + 2 \sin^2 x} = \lim_{x \to 0} \frac{-2 \sin 2x}{2x^2 \cos 2x + 2x \sin 2x + 4x \sin x \cos x + 2 \sin^2 x} = \lim_{x \to 0} \frac{2 \sin 2x}{3 - 2x^2} \sin 2x + 6 \cos 2x = \frac{4 \cos 2x}{3 - 2x^2} \cos 2x + 4 \sin 2x \cos 2x + 6 \sin 2x \cos 2x = -\frac{1}{3}.
\]

35. \[ \lim_{x \to \pi/2} (\sec x - \tan x) \]

**SOLUTION**

\[
\lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2} \frac{1}{\cos x} - \lim_{x \to \pi/2} \frac{\sin x}{\cos x} = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = 0.
\]

36. \[ \lim_{x \to 0} \frac{e^{x^2} - e^4}{x - 2} \]

**SOLUTION**

\[
\lim_{x \to 2} \frac{e^{x^2} - e^4}{x - 2} = \left(2x \frac{e^{x^2}}{x^2} + 2 \frac{e^4}{x} \right) = 4e^4.
\]

37. \[ \lim_{x \to 1} \left( \frac{\pi x}{2} \right) \ln x \]

**SOLUTION**

\[
\lim_{x \to 1} \left( \frac{\pi x}{2} \right) \ln x = \lim_{x \to 1} \frac{\ln x}{\cos \left( \frac{\pi x}{2} \right)} = \lim_{x \to 1} \frac{1}{x} - \frac{\pi}{2} \frac{2x}{\cos^2 \left( \frac{\pi x}{2} \right)} = \lim_{x \to 1} -\frac{2}{\pi x} \sin^2 \left( \frac{\pi x}{2} \right) = -\frac{2}{\pi}.
\]
38. \( \lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x - 1) \ln x} \)

**SOLUTION**

\[
\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x - 1) \ln x} = \lim_{x \to 1} \frac{x(\frac{1}{x}) + (\ln x - 1)}{1 - \frac{1}{x} + \ln x} = \lim_{x \to 1} \frac{\ln x}{1 + \frac{1}{x}} = \frac{1}{1 + 1} = \frac{1}{2}.
\]

39. \( \lim_{x \to 0} \frac{e^x - 1}{\sin x} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \lim_{x \to 0} \frac{e^x}{\cos x} = 1.
\]

40. \( \lim_{x \to 1} \frac{e^x - e}{\ln x} \)

**SOLUTION**

\[
\lim_{x \to 1} \frac{e^x - e}{\ln x} = \lim_{x \to 1} \frac{e^x}{x} = \frac{e}{1} = e.
\]

41. \( \lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x \to 0} \frac{2e^{2x} - 1}{2x} \quad \text{which does not exist.}
\]

42. \( \lim_{x \to \infty} \frac{e^{2x} - 1 - x}{x^2} \)

**SOLUTION**

\[
\lim_{x \to \infty} \frac{e^{2x} - 1 - x}{x^2} = \lim_{x \to \infty} \frac{2e^{2x} - 1}{2x} = \lim_{x \to \infty} \frac{4e^{2x}}{2} = \infty.
\]

43. \( \lim_{t \to 0^+} (\sin t)(\ln t) \)

**SOLUTION**

\[
\lim_{t \to 0^+} (\sin t)(\ln t) = \lim_{t \to 0^+} \frac{\ln t}{\csc t} = \lim_{t \to 0^+} \frac{1}{-\csc t \cot t} = \lim_{t \to 0^+} \frac{-\sin^2 t}{t \cos t} = \lim_{t \to 0^+} \frac{-2 \sin t \cos t}{\cos t - t \sin t} = 0.
\]

44. \( \lim_{x \to \infty} e^{-x}(x^3 - x^2 + 9) \)

**SOLUTION**

\[
\lim_{x \to \infty} e^{-x}(x^3 - x^2 + 9) = \lim_{x \to \infty} \frac{x^3 - x^2 + 9}{e^x} = \lim_{x \to \infty} \frac{3x^2 - 2x}{e^x} = \lim_{x \to \infty} \frac{6x - 2}{e^x} = \lim_{x \to \infty} \frac{6}{e^x} = 0.
\]

45. \( \lim_{x \to 0} \frac{a^x - 1}{x} \quad (a > 0) \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{\ln a \cdot a^x}{1} = \ln a.
\]

46. \( \lim_{x \to \infty} x^{1/x^2} \)

**SOLUTION**

\[
\lim_{x \to \infty} x^{1/x^2} = \lim_{x \to \infty} \frac{\ln x}{x^2} = \lim_{x \to \infty} \frac{1}{2x^2} = 0. \quad \text{Hence,}
\]

\[
\lim_{x \to \infty} x^{1/x^2} = \lim_{x \to \infty} e^{\ln x^{1/x^2}} = e^0 = 1.
\]

47. \( \lim_{x \to 1} (1 + \ln x)^{1/(x-1)} \)

**SOLUTION**

\[
\lim_{x \to 1} \ln(1 + \ln x)^{1/(x-1)} = \lim_{x \to 1} \frac{\ln(1 + \ln x)}{x - 1} = \lim_{x \to 1} \frac{1}{x(1 + \ln x)} = 1. \quad \text{Hence,}
\]

\[
\lim_{x \to 1} (1 + \ln x)^{1/(x-1)} = \lim_{x \to 1} e^{(1 + \ln x)^{1/(x-1)}} = e.
\]
48. \( \lim_{x \to 0^+} x \sin x \)

**SOLUTION**

\[
\lim_{x \to 0^+} x \sin x = \lim_{x \to 0^+} x \cdot \sin x = \lim_{x \to 0^+} \frac{\ln x}{-\cos x} \cdot \frac{\sin x}{\ln x} = \lim_{x \to 0^+} \frac{1}{\cos x} - \cos x = \lim_{x \to 0^+} \frac{2 \sin x \cos x}{-x \sin x + \cos x} = 0.
\]

Hence, \( \lim_{x \to 0^+} x \sin x = \lim_{x \to 0^+} e^{\ln(x \sin x)} = e^0 = 1. \)

49. \( \lim_{x \to 0} (\cos x)^{3/x^2} \)

**SOLUTION**

\[
\lim_{x \to 0} \ln((\cos x)^{3/x^2}) = \lim_{x \to 0} \frac{3 \ln \cos x}{3x^2} = \lim_{x \to 0} -\frac{3 \tan x}{2x} = \lim_{x \to 0} -\frac{3 \sec^2 x}{2x} = -\frac{3}{2}.
\]

Hence, \( \lim_{x \to 0} (\cos x)^{3/x^2} = e^{-3/2} \).

50. \( \lim_{x \to \infty} \left( \frac{x}{x + 1} \right)^x \)

**SOLUTION**

\[
\lim_{x \to \infty} x \ln \left( \frac{x}{x + 1} \right) = \lim_{x \to \infty} \ln \left( \frac{x}{x + 1} \right)^1 = \lim_{x \to \infty} \frac{1}{x} - \frac{1}{x^2} = \lim_{x \to \infty} -\frac{x}{x + 1} = -1.
\]

Hence,

\[
\lim_{x \to \infty} \left( \frac{x}{x + 1} \right)^x = \frac{1}{e}.
\]

51. \( \lim_{x \to 0} \frac{\sin^{-1} x}{x} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{\sin^{-1} x}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1-x^2}} = 1.
\]

52. \( \lim_{x \to 0} \frac{\tan^{-1} x}{\sin^{-1} x} \)

**SOLUTION**

\[
\lim_{x \to 0} \frac{\tan^{-1} x}{\sin^{-1} x} = \lim_{x \to 0} \frac{1}{\frac{1+x^2}{\sqrt{1-x^2}}} = 1.
\]

53. \( \lim_{x \to 1} \frac{\tan^{-1} x - \pi}{\tan x - x} \)

**SOLUTION**

\[
\lim_{x \to 1} \frac{\tan^{-1} x - \pi}{\tan x - x} = \lim_{x \to 1} \frac{1+x^2}{\pi \sec^2(\pi x/4)} = \frac{\pi}{\pi} = 1.
\]

54. \( \lim_{x \to 0^+} \ln x \tan^{-1} x \)

**SOLUTION** Let \( h(x) = \ln x \tan^{-1} x \), \( \lim_{x \to 0} h(x) = -\infty \cdot 0 \), so we apply L'Hôpital's rule to \( h(x) = \frac{f(x)}{g(x)} \), where \( f(x) = \tan^{-1} x \) and \( g(x) = \frac{1}{x} \).

\[
f'(x) = \frac{1}{1+x^2},
\]

\[
\lim_{x \to 0} f'(x) = 1.
\]
\[
g'(x) = -\frac{1}{x(\ln x)^2}
\]

Hence, L'Hôpital's rule yields:

\[
\lim_{x \to 0} g'(x) = -\infty
\]

55. Evaluate \( \lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} \), where \( m, n \neq 0 \) are integers.

**SOLUTION** Suppose \( m \) and \( n \) are even. Then there exist integers \( k \) and \( l \) such that \( m = 2k \) and \( n = 2l \) and

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \frac{\cos k\pi}{\cos l\pi} = (-1)^{k-l}.
\]

Now, suppose \( m \) is even and \( n \) is odd. Then

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx}
\]

does not exist (from one side the limit tends toward \(-\infty\), while from the other side the limit tends toward \(+\infty\)). Third, suppose \( m \) is odd and \( n \) is even. Then

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = 0.
\]

Finally, suppose \( m \) and \( n \) are odd. This is the only case when the limit is indeterminate. Then there exist integers \( k \) and \( l \) such that \( m = 2k + 1, n = 2l + 1 \) and, by L'Hôpital's Rule,

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \lim_{x \to \pi/2} \frac{-m \sin mx}{-n \sin nx} = (-1)^{k-l} \frac{m}{n}.
\]

To summarize,

\[
\lim_{x \to \pi/2} \frac{\cos mx}{\cos nx} = \begin{cases} 
(-1)^{(m-n)/2}, & m, n \text{ even} \\
\text{does not exist}, & m \text{ even, } n \text{ odd} \\
0, & m \text{ odd, } n \text{ even} \\
(-1)^{(m-n)/2} \frac{m}{n}, & m, n \text{ odd}
\end{cases}
\]

56. Evaluate \( \lim_{x \to 1} \frac{x^m - 1}{x^n - 1} \) for any numbers \( m, n \neq 0 \).

**SOLUTION**

\[
\lim_{x \to 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \to 1} \frac{mx^{m-1}}{nx^{n-1}} = \frac{m}{n}.
\]

57. Prove the following limit formula for \( e \):

\[
e = \lim_{x \to 0} (1 + x)^{1/x}
\]

Then find a value of \( x \) such that \(|(1 + x)^{1/x} - e| \leq 0.001\).

**SOLUTION** Using L'Hôpital's Rule,

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1 + x}}{1} = 1.
\]

Thus,

\[
\lim_{x \to 0} \ln \left( (1 + x)^{1/x} \right) = \lim_{x \to 0} \frac{1}{x} \ln(1 + x) = \lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1,
\]

and \( \lim_{x \to 0} (1 + x)^{1/x} = e \). For \( x = 0.0005 \),

\[
| (1 + x)^{1/x} - e | = |(1.0005)^{2000} - e | \approx 6.79 \times 10^{-4} < 0.001.
\]

58. [GU] Can L'Hôpital's Rule be applied to \( \lim_{x \to 0^+} x \sin(1/x) \)? Does a graphical or numerical investigation suggest that the limit exists?
SOLUTION Since \( \sin(1/x) \) oscillates as \( x \to 0^+ \), L’Hôpital’s Rule cannot be applied. Both numerical and graphical investigations suggest that the limit does not exist due to the oscillation.

<table>
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<th>( x )</th>
<th>( 1 )</th>
<th>( 0.1 )</th>
<th>( 0.01 )</th>
<th>( 0.001 )</th>
<th>( 0.0001 )</th>
</tr>
</thead>
<tbody>
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<td>10.2975</td>
<td>0.003316</td>
<td>16.6900</td>
<td>0.6626</td>
</tr>
</tbody>
</table>

59. Let \( f(x) = x^{1/x} \) for \( x > 0 \).

(a) Calculate \( \lim_{x \to 0^+} f(x) \) and \( \lim_{x \to \infty} f(x) \).

(b) Find the maximum value of \( f(x) \), and determine the intervals on which \( f(x) \) is increasing or decreasing.

SOLUTION

(a) Let \( f(x) = x^{1/x} \). Note that \( \lim_{x \to 0^+} x^{1/x} \) is not indeterminate. As \( x \to 0^+ \), the base of the function tends toward 0 and the exponent tends toward \( +\infty \). Both of these factors force \( x^{1/x} \) toward 0. Thus, \( \lim_{x \to 0^+} f(x) = 0 \). On the other hand, \( \lim_{x \to \infty} f(x) \) is indeterminate. We calculate this limit as follows:

\[
\lim_{x \to \infty} \ln f(x) = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0.
\]

so \( \lim_{x \to \infty} f(x) = e^0 = 1 \).

(b) Again, let \( f(x) = x^{1/x} \), so that \( \ln f(x) = \frac{1}{x} \ln x \). To find the derivative \( f' \), we apply the derivative to both sides:

\[
\frac{d}{dx} \ln f(x) = \frac{\ln x}{x^2} + \frac{1}{x} = \frac{x^{1/x}}{x^2} (1 - \ln x)
\]

Thus, \( f \) is increasing for \( 0 < x < e \), is decreasing for \( x > e \) and has a maximum at \( x = e \). The maximum value is \( f(e) = e^{1/e} \approx 1.444668 \).

60. (a) Use the results of Exercise 59 to prove that \( x^{1/x} = c \) has a unique solution if \( 0 < c \leq 1 \) or \( c = e^{1/e} \), two solutions if \( 1 < c < e^{1/e} \), and no solutions if \( c > e^{1/e} \).

(b) Plot the graph of \( f(x) = x^{1/x} \) and verify that it confirms the conclusions of (a).

SOLUTION

(a) Because \( (e, e^{1/e}) \) is the only maximum, no solution exists for \( c > e^{1/e} \) and only one solution exists for \( c = e^{1/e} \). Moreover, because \( f(x) \) increases from 0 to \( e^{1/e} \) as \( x \) goes from 0 to \( e \) and then decreases from \( e^{1/e} \) to 1 as \( x \) goes from \( e \) to \( +\infty \), it follows that there are two solutions for \( 1 < c < e^{1/e} \), but only one solution for 0 < \( c \leq 1 \).

(b) Observe that if we sketch the horizontal line \( y = c \), this line will intersect the graph of \( y = f(x) \) only once for \( 0 < c \leq 1 \) and \( c = e^{1/e} \) and will intersect the graph of \( y = f(x) \) twice for \( 1 < c < e^{1/e} \). There are no points of intersection for \( c > e^{1/e} \).

61. Determine whether \( f << g \) or \( g << f \) (or neither) for the functions \( f(x) = \log_{10} x \) and \( g(x) = \ln x \).
SOLUTION Because

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{\log_{10} x}{\ln x} = \lim_{x \to \infty} \frac{\ln x}{\ln 10} = 1. \]

neither \( f \ll g \) or \( g \ll f \) is satisfied.

62. Show that \((\ln x)^3 \ll \sqrt{x}\) and \((\ln x)^4 \ll x^{1/10}\).

SOLUTION

- \((\ln x)^3 \ll \sqrt{x}\):

\[ \lim_{x \to \infty} \frac{\sqrt{x}}{(\ln x)^3} = \lim_{x \to \infty} \frac{1}{\frac{3}{2} \ln x} = \lim_{x \to \infty} \frac{\sqrt{x}}{4 \ln x} = \lim_{x \to \infty} \frac{1}{\frac{4}{2}} = \lim_{x \to \infty} \frac{\sqrt{x}}{8} = \infty. \]

- \((\ln x)^4 \ll x^{1/10}\):

\[ \lim_{x \to \infty} \frac{x^{1/10}}{(\ln x)^4} = \lim_{x \to \infty} \frac{\frac{1}{10}x^{9/10}}{(\ln x)^3} = \lim_{x \to \infty} \frac{x^{1/10}}{120(\ln x)^2} = \lim_{x \to \infty} \frac{x^{1/10}}{240000} = \infty. \]

63. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that \( \ln x \ll x^a \) for all \( a > 0 \).

SOLUTION Using L'Hôpital's Rule:

\[ \lim_{x \to \infty} \frac{\ln x}{x^a} = \lim_{x \to \infty} x^{-1} a x^{a-1} = \lim_{x \to \infty} a x^{-a} = 0; \]

hence, \( \ln x \ll x^a \).

64. Show that \((\ln x)^N \ll x^a\) for all \( N \) and all \( a > 0 \).

SOLUTION

\[ \lim_{x \to \infty} \frac{x^a}{(\ln x)^N} = \lim_{x \to \infty} \frac{a x^{a-1}}{N(\ln x)^{N-1}} = \lim_{x \to \infty} a x^a/N(\ln x)^{N-1} = \cdots \]

If we continue in this manner, L'Hôpital's Rule will give a factor of \( x^a \) in the numerator, but the power on \( \ln x \) in the denominator will eventually be zero. Thus,

\[ \lim_{x \to \infty} \frac{x^a}{(\ln x)^N} = 0, \]

so \((\ln x)^N \ll x^a\) for all \( N \) and for all \( a > 0 \).

65. Determine whether \( \sqrt{x} \ll e^{\sqrt{\ln x}} \) or \( e^{\sqrt{\ln x}} \ll \sqrt{x} \). Hint: Use the substitution \( u = \ln x \) instead of L'Hôpital's Rule.

SOLUTION Let \( u = \ln x \), then \( x = e^u \), and as \( x \to \infty \), \( u \to \infty \). So

\[ \lim_{x \to \infty} e^{\sqrt{\ln x}} = \lim_{u \to \infty} e^{\sqrt{u}} = \lim_{u \to \infty} e^{\sqrt{u} - u/2}. \]

We need to examine \( \lim_{u \to \infty} (\sqrt{u} - u/2) \). Since

\[ \lim_{u \to \infty} u/2 \sqrt{u} = \lim_{u \to \infty} \frac{1}{2} u = \lim_{u \to \infty} \sqrt{u} = \infty, \]

\[ \sqrt{u} = O(u/2) \text{ and } \lim_{u \to \infty} \left( \sqrt{u} - \frac{u}{2} \right) = -\infty. \] Thus

\[ \lim_{u \to \infty} e^{\sqrt{u} - u/2} = e^{-\infty} = 0 \text{ and } \lim_{x \to \infty} e^{\sqrt{\ln x}} = 0 \]

and \( e^{\sqrt{\ln x}} \ll \sqrt{x} \).
66. Show that \( \lim_{x \to \infty} x^n e^{-x} = 0 \) for all whole numbers \( n > 0 \).

**SOLUTION**

\[
\lim_{x \to \infty} x^n e^{-x} = \lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{n x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{n(n-1) x^{n-2}}{e^x} \]

\[
\vdots
\]

\[
= \lim_{x \to \infty} \frac{n!}{e^x} = 0.
\]

67. Assumptions Matter

Let \( f(x) = x(2 + \sin x) \) and \( g(x) = x^2 + 1 \).

(a) Show directly that \( \lim_{x \to \infty} f(x)/g(x) = 0 \).

(b) Show that \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \), but \( \lim_{x \to \infty} f'(x)/g'(x) \) does not exist.

Do (a) and (b) contradict L’Hôpital’s Rule? Explain.

**SOLUTION**

(a) \( 1 \leq 2 + \sin x \leq 3 \), so

\[
\frac{x}{x^2 + 1} \leq \frac{x(2 + \sin x)}{x^2 + 1} \leq \frac{3x}{x^2 + 1}.
\]

Since,

\[
\lim_{x \to \infty} \frac{x}{x^2 + 1} = \lim_{x \to \infty} \frac{3x}{x^2 + 1} = 0,
\]

it follows by the Squeeze Theorem that

\[
\lim_{x \to \infty} \frac{x(2 + \sin x)}{x^2 + 1} = 0.
\]

(b) \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} x(2 + \sin x) \geq \lim_{x \to \infty} x = \infty \) and \( \lim_{x \to \infty} g(x) = \lim_{x \to \infty} (x^2 + 1) = \infty \), but

\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{x(\cos x) + (2 + \sin x)}{2x}
\]

does not exist since \( \cos x \) oscillates. This does not violate L’Hôpital’s Rule since the theorem clearly states

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\]

“provided the limit on the right exists.”

68. Let \( H(b) = \lim_{x \to \infty} \frac{\ln(1 + b^x)}{x} \) for \( b > 0 \).

(a) Show that \( H(b) = \ln b \) if \( b \geq 1 \)

(b) Determine \( H(b) \) for \( 0 < b < 1 \).

**SOLUTION**

(a) Suppose \( b \geq 1 \). Then

\[
H(b) = \lim_{x \to \infty} \frac{\ln(1 + b^x)}{x} = \lim_{x \to \infty} \frac{b^x \ln b}{1 + b^x} = \frac{b^x \ln b}{b^x} = \ln b.
\]

(b) Now, suppose \( 0 < b < 1 \). Then

\[
H(b) = \lim_{x \to \infty} \frac{\ln(1 + b^x)}{x} = \lim_{x \to \infty} \frac{b^x \ln b}{1 + b^x} = \frac{0}{1} = 0.
\]

69. Let \( G(b) = \lim_{x \to \infty} (1 + b^x)^{1/x} \).

(a) Use the result of Exercise 68 to evaluate \( G(b) \) for all \( b > 0 \).

(b) Verify your result graphically by plotting \( y = (1 + b^x)^{1/x} \) together with the horizontal line \( y = G(b) \) for the values \( b = 0.25, 0.5, 2, 3 \).
(a) Using Exercise 68, we see that \(G(b) = e^{H(b)}\). Thus, \(G(b) = 1\) if \(0 \leq b \leq 1\) and \(G(b) = b\) if \(b > 1\).

(b) 

70. Show that \(\lim_{t \to \infty} t^k e^{-t^2} = 0\) for all \(k\). *Hint:* Compare with \(\lim_{t \to \infty} t^k e^{-t} = 0\).

**SOLUTION**  
Because we are interested in the limit as \(t \to +\infty\), we will restrict attention to \(t > 1\). Then, for all \(k\),

\[0 \leq t^k e^{-t^2} \leq t^k e^{-t}.
\]

As \(\lim_{t \to \infty} t^k e^{-t} = 0\), it follows from the Squeeze Theorem that

\[\lim_{t \to \infty} t^k e^{-t^2} = 0.
\]

In Exercises 71–73, let

\[f(x) = \begin{cases} 
  e^{-1/x^2} & \text{for } x \neq 0 \\
  0 & \text{for } x = 0
\end{cases}
\]

These exercises show that \(f(x)\) has an unusual property: All of its derivatives at \(x = 0\) exist and are equal to zero.

71. Show that \(\lim_{x \to 0} f(x) = 0\) for all \(k\). *Hint:* Let \(t = x^{-1}\) and apply the result of Exercise 70.

**SOLUTION**  
\(\lim_{x \to 0} \frac{f(x)}{x^k} = \lim_{x \to 0} \frac{1}{x^k e^{1/x^2}}\). Let \(t = 1/x\). As \(x \to 0\), \( t \to \infty\). Thus,

\[\lim_{x \to 0} \frac{1}{x^k e^{1/x^2}} = \lim_{t \to \infty} t^k e^{-t^2} = 0
\]

by Exercise 70.

72. Show that \(f'(0)\) exists and is equal to zero. Also, verify that \(f''(0)\) exists and is equal to zero.

**SOLUTION**  
Working from the definition,

\[f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = 0
\]

by the previous exercise. Thus, \(f'(0)\) exists and is equal to 0. Moreover,

\[f'(x) = \begin{cases} 
  e^{-1/x^2} \left(\frac{2}{x^4}\right) & \text{for } x \neq 0 \\
  0 & \text{for } x = 0
\end{cases}
\]

Now,

\[f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} e^{-1/x^2} \left(\frac{2}{x^4}\right) = 2 \lim_{x \to 0} \frac{f(x)}{x^4} = 0
\]

by the previous exercise. Thus, \(f''(0)\) exists and is equal to 0.
73. Show that for \( k \geq 1 \) and \( x \neq 0 \),

\[
f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}
\]

for some polynomial \( P(x) \) and some exponent \( r \geq 1 \). Use the result of Exercise 71 to show that \( f^{(k)}(0) \) exists and is equal to zero for all \( k \geq 1 \).

**SOLUTION**

For \( x \neq 0 \), \( f'(x) = e^{-1/x^2} \left( \frac{2}{x^3} \right) \). Here \( P(x) = 2 \) and \( r = 3 \). Assume \( f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r} \). Then

\[
f^{(k+1)}(x) = e^{-1/x^2} \left( \frac{x^3f'(x) + (2 - rx^2)f(x)}{x^{r+3}} \right)
\]

which is of the form desired.

Moreover, from Exercise 72, \( f'(0) = 0 \). Suppose \( f^{(k)}(0) = 0 \). Then

\[
f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x \to 0} \frac{P(x)e^{-1/x^2}}{x^{r+1}} = P(0) \lim_{x \to 0} \frac{f(x)}{x^{r+1}} = 0.
\]

### Further Insights and Challenges

74. Show that L'Hôpital's Rule applies to \( \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} \) but that it does not help. Then evaluate the limit directly.

**SOLUTION**

Both the numerator \( f(x) = x \) and the denominator \( g(x) = \sqrt{x^2 + 1} \) tend to infinity as \( x \to \infty \), and \( g'(x) = x/\sqrt{x^2 + 1} \) is nonzero for \( x > 0 \). Therefore, L'Hôpital's Rule applies:

\[
\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{(x^2 + 1)^{-1/2}} = \lim_{x \to \infty} \frac{(x^2 + 1)^{1/2}}{x}
\]

We may apply L'Hôpital's Rule again:

\[
\lim_{x \to \infty} \frac{(x^2 + 1)^{1/2}}{x} = \lim_{x \to \infty} \frac{x^2 + 1}{1} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}.
\]

This takes us back to the original limit, so L'Hôpital's Rule is ineffective. However, we can evaluate the limit directly by observing that

\[
\frac{x}{\sqrt{x^2 + 1}} = \frac{x^{-1}(x)}{\sqrt{1 + x^{-2}}} \quad \text{and hence} \quad \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + x^{-2}}} = 1.
\]

75. The Second Derivative Test for critical points fails if \( f''(c) = 0 \). This exercise develops a Higher Derivative Test based on the sign of the first nonzero derivative. Suppose that

\[
f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0
\]

(a) Show, by applying L'Hôpital's Rule \( n \) times, that

\[
\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^n} = \frac{1}{n!} f^{(n)}(c)
\]

where \( n! = n(n-1)(n-2) \cdots (2)(1) \).

(b) Use (a) to show that if \( n \) is even, then \( f(c) \) is a local minimum if \( f^{(n)}(c) > 0 \) and is a local maximum if \( f^{(n)}(c) < 0 \). *Hint:* If \( n \) is even, then \( (x - c)^n > 0 \) for \( x \neq a \), so \( f(x) - f(c) \) must be positive for \( x \) near \( c \) if \( f^{(n)}(c) > 0 \).

(c) Use (a) to show that if \( n \) is odd, then \( f(c) \) is neither a local minimum nor a local maximum.

**SOLUTION**

(a) Repeated application of L'Hôpital's rule yields

\[
\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^n} = \lim_{x \to c} \frac{f'(x)}{n(x - c)^{n-1}}
\]

\[
= \lim_{x \to c} \frac{f''(x)}{n(n-1)(x - c)^{n-2}}
\]

\[
= \lim_{x \to c} \frac{f'''(x)}{n(n-1)(n-2)(x - c)^{n-3}}
\]

\[
\cdots
\]

\[
= \frac{1}{n!} f^{(n)}(c)
\]
(b) Suppose \( n \) is even. Then \((x - c)^n > 0\) for all \( x \neq c \). If \( f^{(n)}(c) > 0 \), it follows that \( f(x) - f(c) \) must be positive for \( x \) near \( c \). In other words, \( f(x) > f(c) \) for \( x \) near \( c \) and \( f(c) \) is a local minimum. On the other hand, if \( f^{(n)}(c) < 0 \), it follows that \( f(x) - f(c) \) must be negative for \( x \) near \( c \). In other words, \( f(x) < f(c) \) for \( x \) near \( c \) and \( f(c) \) is a local maximum.

(c) If \( n \) is odd, then \((x - c)^n > 0\) for \( x > c \) but \((x - c)^n < 0\) for \( x < c \). If \( f^{(n)}(c) > 0 \), it follows that \( f(x) - f(c) \) must be positive for \( x \) near \( c \) and \( x > c \) but is negative for \( x \) near \( c \) and \( x < c \). In other words, \( f(x) > f(c) \) for \( x \) near \( c \) and \( x > c \) but \( f(x) < f(c) \) for \( x \) near \( c \) and \( x < c \). Thus, \( f(c) \) is neither a local minimum nor a local maximum. We obtain a similar result if \( f^{(n)}(c) < 0 \).

76. When a spring with natural frequency \( \lambda/2\pi \) is driven with a sinusoidal force \( \sin(\omega t) \) with \( \omega \neq \lambda \), it oscillates according to

\[ y(t) = \frac{1}{\lambda^2 - \omega^2}(\lambda \sin(\omega t) - \omega \sin(\lambda t)) \]

Let \( y_0(t) = \lim_{\omega \to \lambda} y(t) \).

(a) Use L'Hôpital's Rule to determine \( y_0(t) \).

(b) Show that \( y_0(t) \) ceases to be periodic and that its amplitude \( |y_0(t)| \) tends to \( \infty \) as \( t \to \infty \) (the system is said to be in resonance; eventually, the spring is stretched beyond its limits).

(c) \( \mathcal{R} \) \( \mathcal{A} \) \( \mathcal{S} \) Plot \( y(t) \) for \( \lambda = 1 \) and \( \omega = 0.8, 0.9, 0.99, \) and 0.999. Do the graphs confirm your conclusion in (b)?

**Solution**

(a)

\[ \lim_{\omega \to \lambda} y(t) = \lim_{\omega \to \lambda} \frac{\lambda \sin(\omega t) - \omega \sin(\lambda t)}{\lambda^2 - \omega^2} = \frac{d}{d\omega}(\lambda \sin(\omega t) - \omega \sin(\lambda t)) \]

\[ = \lim_{\omega \to \lambda} \frac{\lambda t \cos(\omega t) - \sin(\lambda t)}{-2\omega} = \frac{\lambda t \cos(\lambda t) - \sin(\lambda t)}{-2\lambda} \]

(b) From part (a)

\[ y_0(t) = \lim_{\omega \to \lambda} y(t) = \frac{\lambda t \cos(\lambda t) - \sin(\lambda t)}{-2\lambda}. \]

This may be rewritten as

\[ y_0(t) = \frac{\sqrt{\lambda^2 t^2 + 1} \cos(\lambda t + \phi)}{-2\lambda}, \]

where \( \cos \phi = \frac{\lambda t}{\sqrt{\lambda^2 t^2 + 1}} \) and \( \sin \phi = \frac{1}{\sqrt{\lambda^2 t^2 + 1}} \). Since the amplitude varies with \( t \), \( y_0(t) \) is not periodic. Also note that

\[ \frac{\sqrt{\lambda^2 t^2 + 1}}{-2\lambda} \to \infty \quad \text{as} \quad t \to \infty. \]

(c) The graphs below were produced with \( \lambda = 1 \). Moving from left to right and from top to bottom, \( \omega = 0.5, 0.8, 0.9, 0.99, 0.999, 1 \).

77. We expended a lot of effort to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \) in Chapter 2. Show that we could have evaluated it easily using L'Hôpital’s Rule. Then explain why this method would involve circular reasoning.

**Solution**

\[ \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1. \] To use L'Hôpital’s Rule to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \), we must know that the derivative of \( \sin x \) is \( \cos x \), but to determine the derivative of \( \sin x \), we must be able to evaluate \( \lim_{x \to 0} \frac{\sin x}{x} \).
78. By a fact from algebra, if \( f, g \) are polynomials such that \( f(a) = g(a) = 0 \), then there are polynomials \( f_1, g_1 \) such that

\[
f(x) = (x - a)f_1(x), \quad g(x) = (x - a)g_1(x)
\]

Use this to verify L'Hôpital's Rule directly for \( \lim_{x \to a} f(x)/g(x) \).

**SOLUTION**  As in the problem statement, let \( f(x) \) and \( g(x) \) be two polynomials such that \( f(a) = g(a) = 0 \), and let \( f_1(x) \) and \( g_1(x) \) be the polynomials such that \( f(x) = (x - a)f_1(x) \) and \( g(x) = (x - a)g_1(x) \). By the product rule, we have the following facts,

\[
f'(x) = (x - a)f_1'(x) + f_1(x) \\
g'(x) = (x - a)g_1'(x) + g_1(x)
\]

so

\[
\lim_{x \to a} f'(x) = f_1(a) \quad \text{and} \quad \lim_{x \to a} g'(x) = g_1(a).
\]

L'Hôpital's Rule stated for \( f \) and \( g \) is: if \( \lim_{x \to a} g'(x) \neq 0 \), so that \( g_1(a) \neq 0 \),

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f_1(a)}{g_1(a)}
\]

Suppose \( g_1(a) \neq 0 \). Then, by direct computation,

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{(x - a)f_1(x)}{(x - a)g_1(x)} = \lim_{x \to a} \frac{f_1(x)}{g_1(x)} = \frac{f_1(a)}{g_1(a)}
\]

exactly as predicted by L'Hôpital's Rule.

79. **Patience Required**  Use L'Hôpital's Rule to evaluate and check your answers numerically:

(a) \( \lim_{x \to 0^+} \left( \frac{\sin x}{x} \right)^{1/x^2} \)

(b) \( \lim_{x \to 0} \left( \frac{1}{\sin^2 x} - \frac{1}{x^2} \right) \)

**SOLUTION**

(a) We start by evaluating

\[
\lim_{x \to 0^+} \ln \left( \frac{\sin x}{x} \right)^{1/x^2} = \lim_{x \to 0^+} \frac{\ln(\sin x) - \ln x}{x^2}.
\]

Repeatedly using L'Hôpital's Rule, we find

\[
\lim_{x \to 0^+} \ln \left( \frac{\sin x}{x} \right)^{1/x^2} = \lim_{x \to 0^+} \frac{-x \sin x}{2x^2 \cos x + 4x \sin x} = \lim_{x \to 0^+} \frac{-2x \cos x + x \sin x}{12 \cos x - 2 \cos^2 x - 12 \sin x}
\]

\[
= \frac{2}{12} = \frac{1}{6}
\]

Therefore, \( \lim_{x \to 0^+} \left( \frac{\sin x}{x} \right)^{1/x^2} = e^{-1/6} \). Numerically we find:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{\sin x}{x} )</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-1/6} )</td>
<td>0.844471</td>
<td>0.846435</td>
<td>0.846481</td>
<td></td>
</tr>
</tbody>
</table>

Note that \( e^{-1/6} \approx 0.846481724 \).

(b) Repeatedly using L'Hôpital's Rule and simplifying, we find

\[
\lim_{x \to 0} \left( \frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x \to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \to 0} \frac{2x - 2 \sin x \cos x}{2 \sin^2 x + 2 \sin^2 x} = \lim_{x \to 0} \frac{2x - 2 \sin 2x}{2 \cos 2x}
\]

\[
= \lim_{x \to 0} \frac{2 \cos 2x + 2x \sin 2x + 4x \sin x \cos x + 2 \sin^2 x}{2 \cos 2x}
\]

\[
= \lim_{x \to 0} \frac{2 \cos 2x}{2 \cos 2x + 4x \sin 2x + 2 \sin^2 x}
\]
Thus, \( f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \) (\( c = 1 \))

(b) \( f(x) = x^6 - x^3 \) (\( c = 0 \))

**SOLUTION**

(a) Let \( f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \). Then \( f'(x) = 5x^4 - 24x^3 + 42x^2 - 32x + 9 \), so \( f'(1) = 5 - 24 + 42 - 32 + 9 = 0 \) and \( c = 1 \) is a critical point. Now,

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= 20x^3 - 72x^2 + 84x - 32 \\
\frac{d^3}{dx^3} f(x) &= 60x^2 - 144x + 84 \\
\frac{d^4}{dx^4} f(x) &= 120x - 144 \\
n &= 4 \\
n &= 4 - 4 \\
&= 0 \neq 0.
\end{align*}
\]

Thus, \( n = 4 \) is even and \( f^{(4)}(x) < 0 \), so \( f(1) \) is a local maximum.

(b) Let \( f(x) = x^6 - x^3 \). Then, \( f'(x) = 6x^5 - 3x^2 \), so \( f'(0) = 0 \) and \( c = 0 \) is a critical point. Now,

\[
\begin{align*}
\frac{d^2}{dx^2} f(x) &= 30x^4 - 6x \\
\frac{d^3}{dx^3} f(x) &= 120x - 6 \\
n &= 3 \\
n &= 3 - 3 \\
&= 0 \neq 0.
\end{align*}
\]

Thus, \( n = 3 \) is odd, so \( f(0) \) is neither a local minimum nor a local maximum.

### 4.6 Graph Sketching and Asymptotes

**Preliminary Questions**

1. Sketch an arc where \( f' \) and \( f'' \) have the sign combination ++. Do the same for --.

**SOLUTION** An arc with the sign combination ++ (increasing, concave up) is shown below at the left. An arc with the sign combination -- (decreasing, concave up) is shown below at the right.

![Graph Sketching](image)

2. If the sign combination of \( f' \) and \( f'' \) changes from ++ to -- at \( x = c \), then (choose the correct answer):
   (a) \( f(c) \) is a local min
   (b) \( f(c) \) is a local max
   (c) \( c \) is a point of inflection

**SOLUTION** Because the sign of the second derivative changes at \( x = c \), the correct response is (c): \( c \) is a point of inflection.

3. The second derivative of the function \( f(x) = (x - 4)^{-1} \) is \( f''(x) = 2(x - 4)^{-3} \). Although \( f''(x) \) changes sign at \( x = 4 \), \( f(x) \) does not have a point of inflection at \( x = 4 \). Why not?

**SOLUTION** The function \( f \) does not have a point of inflection at \( x = 4 \) because \( x = 4 \) is not in the domain of \( f \).