(c) Substitute \( u = g(x) \) in Eq. (2) to obtain
\[
\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x)) \frac{g(x) - g(a)}{x - a}
\]
Derive the Chain Rule by computing the limit of both sides as \( x \to a \).

**SOLUTION** For any differentiable function \( f \) and any number \( b \), define
\[
F(u) = \frac{f(u) - f(b)}{u - b}
\]
for all \( u \neq b \).

(a) Define \( F(b) = f'(b) \). Then
\[
\lim_{u \to b} F(u) = \lim_{u \to b} \frac{f(u) - f(b)}{u - b} = f'(b) = F(b),
\]
i.e., \( \lim_{u \to b} F(u) = F(b) \). Therefore, \( F \) is continuous at \( u = b \).

(b) Let \( g \) be a differentiable function and take \( b = g(a) \). Let \( x \) be a number distinct from \( a \). If we substitute \( u = g(a) \) into Eq. (2), both sides evaluate to 0, so equality is satisfied. On the other hand, if \( u \neq g(a) \), then
\[
\frac{f(u) - f(g(a))}{x - a} = \frac{f(u) - f(g(a))}{u - g(a)} \frac{u - g(a)}{x - a} = \frac{f(u) - f(b)}{u - b} \frac{u - g(a)}{x - a} = F(u) \frac{u - g(a)}{x - a}
\]
Hence for all \( u \), we have
\[
\frac{f(u) - f(g(a))}{x - a} = F(u) \frac{u - g(a)}{x - a}.
\]

(c) Substituting \( u = g(x) \) in Eq. (2), we have
\[
\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x)) \frac{g(x) - g(a)}{x - a}
\]
Letting \( x \to a \) gives
\[
\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x \to a} \left( F(g(x)) \frac{g(x) - g(a)}{x - a} \right) = F(g(a))g'(a) = F(b)g'(a) = f'(b)g'(a)
\]
Therefore \( (f \circ g)'(a) = f'(g(a))g'(a) \), which is the Chain Rule.

### 3.8 Derivatives of Inverse Functions

#### Preliminary Questions

1. What is the slope of the line obtained by reflecting the line \( y = \frac{1}{2}x \) through the line \( y = x \)?

**SOLUTION** The line obtained by reflecting the line \( y = x/2 \) through the line \( y = x \) has slope 2.

2. Suppose that \( P = (2, 4) \) lies on the graph of \( f(x) \) and that the slope of the tangent line through \( P \) is \( m = 3 \). Assuming that \( f^{-1}(x) \) exists, what is the slope of the tangent line to the graph of \( f^{-1}(x) \) at the point \( Q = (4, 2) \)?

**SOLUTION** The tangent line to the graph of \( f^{-1}(x) \) at the point \( Q = (4, 2) \) has slope \( \frac{1}{3} \).

3. Which inverse trigonometric function \( g(x) \) has the derivative \( g'(x) = \frac{1}{x^2 + 1} \)?

**SOLUTION** \( g(x) = \tan^{-1} x \) has the derivative \( g'(x) = \frac{1}{x^2 + 1} \).

4. What does the following identity tell us about the derivatives of \( \sin^{-1} x \) and \( \cos^{-1} x \)?

\[
\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}
\]

**SOLUTION** Angles whose sine and cosine are \( x \) are complementary.
Exercises

1. Find the inverse \( g(x) \) of \( f(x) = \sqrt{x^2 + 9} \) with domain \( x \geq 0 \) and calculate \( g'(x) \) in two ways: using Theorem 1 and by direct calculation.

**Solution** To find a formula for \( g(x) = f^{-1}(x) \), solve \( y = \sqrt{x^2 + 9} \) for \( x \). This yields \( x = \pm \sqrt{y^2 - 9} \). Because the domain of \( f \) was restricted to \( x \geq 0 \), we must choose the positive sign in front of the radical. Thus

\[
g(x) = f^{-1}(x) = \sqrt{x^2 - 9}.
\]

Because \( x^2 + 9 \geq 9 \) for all \( x \), it follows that \( f(x) \geq 3 \) for all \( x \). Thus, the domain of \( g(x) = f^{-1}(x) \) is \( x \geq 3 \). The range of \( g \) is the restricted domain of \( f \): \( y \geq 0 \).

By Theorem 1,

\[
g'(x) = \frac{1}{f'(g(x))}.
\]

With

\[
f'(x) = \frac{x}{\sqrt{x^2 + 9}},
\]

it follows that

\[
f'(g(x)) = \frac{\sqrt{x^2 - 9}}{\sqrt{(x^2 - 9)^2 + 9}} = \frac{\sqrt{x^2 - 9}}{\sqrt{x^2}} = \frac{\sqrt{x^2 - 9}}{x}
\]

since the domain of \( g \) is \( x \geq 3 \). Thus,

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{x}{\sqrt{x^2 - 9}}.
\]

This agrees with the answer we obtain by differentiating directly:

\[
g'(x) = \frac{2x}{2\sqrt{x^2 - 9}} = \frac{x}{\sqrt{x^2 - 9}}.
\]

2. Let \( g(x) \) be the inverse of \( f(x) = x^3 + 1 \). Find a formula for \( g(x) \) and calculate \( g'(x) \) in two ways: using Theorem 1 and then by direct calculation.

**Solution** To find \( g(x) \), we solve \( y = x^3 + 1 \) for \( x \):

\[
y - 1 = x^3
\]

\[
x = (y - 1)^{1/3}
\]

Therefore, the inverse is \( g(x) = (x - 1)^{1/3} \).

We have \( f'(x) = 3x^2 \). According to Theorem 1,

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3(x - 1)^{2/3}} = \frac{1}{3} (x - 1)^{-2/3}
\]

This agrees with the answer we obtain by differentiating directly:

\[
d dx (x - 1)^{1/3} = \frac{1}{3} (x - 1)^{-2/3}.
\]

In Exercises 3–8, use Theorem 1 to calculate \( g'(x) \), where \( g(x) \) is the inverse of \( f(x) \).

3. \( f(x) = 7x + 6 \)

**Solution** Let \( f(x) = 7x + 6 \) then \( f'(x) = 7 \). Solving \( y = 7x + 6 \) for \( x \) and switching variables, we obtain the inverse \( g(x) = (x - 6)/7 \). Thus,

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{7}.
\]

4. \( f(x) = \sqrt{3 - x} \)
SOLUTION Let \( f(x) = (3 - x)^{1/2} \). Then

\[
f'(x) = \frac{1}{2}(3 - x)^{-1/2}(-1) = -\frac{1}{2(3 - x)^{1/2}}.\]

Solving \( y = \sqrt{3 - x} \) for \( x \) and switching variables, we obtain the inverse \( g(x) = 3 - x^2 \). Thus,

\[
g'(x) = \frac{1}{2(3 - x^2)^{1/2}} = -2x,
\]

where we have used the fact that the domain of \( g \) is \( x \geq 0 \) to write \( \sqrt{x^2} = x \).

5. \( f(x) = x^{-5} \)

SOLUTION Let \( f(x) = x^{-5} \), then \( f'(x) = -5x^{-6} \). Solving \( y = x^{-5} \) for \( x \) and switching variables, we obtain the inverse \( g(x) = x^{-1/5} \). Thus,

\[
g'(x) = \frac{1}{-5x^{-6/5}} = -\frac{1}{5}x^{-6/5}.
\]

6. \( f(x) = 4x^3 - 1 \)

SOLUTION Let \( f(x) = 4x^3 - 1 \), then \( f'(x) = 12x^2 \). Solving \( y = 4x^3 - 1 \) for \( x \) and switching variables, we obtain the inverse \( g(x) = (\frac{x+1}{4})^{1/3} \). Thus,

\[
g'(x) = \frac{1}{12} \left( \frac{x+1}{4} \right)^{-2/3}
\]

7. \( f(x) = \frac{x}{x + 1} \)

SOLUTION Let \( f(x) = \frac{x}{x + 1} \), then

\[
f'(x) = \frac{(x + 1) - x}{(x + 1)^2} = \frac{1}{(x + 1)^2}.
\]

Solving \( y = \frac{x}{x + 1} \) for \( x \) and switching variables, we obtain the inverse \( g(x) = \frac{x}{1-x} \). Thus

\[
g'(x) = 1\left/ \frac{-1}{(1-x)^2} \right. = \frac{1}{1-x^2}.
\]

8. \( f(x) = 2 + x^{-1} \)

SOLUTION Let \( f(x) = 2 + x^{-1} \), then \( f'(x) = -1/x^2 \). Solving \( y = 2 + x^{-1} \) for \( x \) and switching variables, we obtain the inverse \( g(x) = 1/(x - 2) \). Thus,

\[
g'(x) = \frac{-1}{1/(x - 2)^2} = \frac{-1}{x - 2^2}.
\]

9. Let \( g(x) \) be the inverse of \( f(x) = x^3 + 2x + 4 \). Calculate \( g'(7) \) [without finding a formula for \( g(x) \)], and then calculate \( g'(7) \).

SOLUTION Let \( g(x) \) be the inverse of \( f(x) = x^3 + 2x + 4 \). Because

\[
f(1) = 1^3 + 2(1) + 4 = 7,
\]

it follows that \( g(7) = 1 \). Moreover, \( f'(x) = 3x^2 + 2 \), and

\[
g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.
\]

10. Find \( g'(\frac{1}{2}) \), where \( g(x) \) is the inverse of \( f(x) = \frac{x^3}{x^2 + 1} \).

SOLUTION Let \( g(x) \) be the inverse of \( f(x) = \frac{x^3}{x^2 + 1} \). Because

\[
f(-1) = \frac{(-1)^3}{(-1)^2 + 1} = \frac{-1}{2},
\]

it follows that \( g(-\frac{1}{2}) = -1 \). Moreover,

\[
f'(x) = \frac{(x^2 + 1)(3x^2) - x^3(2x)}{(x^2 + 1)^2} = \frac{x^4 + 3x^2}{(x^2 + 1)^2}
\]

and

\[
g'\left(\frac{1}{2}\right) = \frac{1}{f'(g(-\frac{1}{2}))} = \frac{1}{f'(-1)} = 1.
\]
In Exercises 11–16, calculate \( g(b) \) and \( g'(b) \), where \( g \) is the inverse of \( f \) (in the given domain, if indicated).

11. \( f(x) = x + \cos x \), \( b = 1 \)
   **Solution** \( f(0) = 1 \), so \( g(1) = 0 \). \( f'(x) = 1 - \sin x \) so \( f'(g(1)) = f'(0) = 1 - \sin 0 = 1 \). Thus, \( g'(1) = 1/1 = 1 \).

12. \( f(x) = 4x^3 - 2x \), \( b = -2 \)
   **Solution** \( f(-1) = -2 \), so \( g(-2) = -1 \). \( f'(x) = 12x^2 - 2 \) so \( f'(g(-2)) = f'(-1) = 12 - 2 = 10 \). Thus, \( g'(-2) = 1/10 \).

13. \( f(x) = \sqrt{x^2 + 6x} \) for \( x \geq 0 \), \( b = 4 \)
   **Solution** To determine \( g(4) \), we solve \( f(x) = \sqrt{x^2 + 6x} = 4 \) for \( x \). This yields:
   \[
   \begin{align*}
   x^2 + 6x &= 16 \\
   x^2 + 6x - 16 &= 0 \\
   (x + 8)(x - 2) &= 0
   \end{align*}
   \]
   or \( x = -8, 2 \). Because the domain of \( f \) has been restricted to \( x \geq 0 \), we have \( g(4) = 2 \). With
   \[
   f'(x) = \frac{x + 3}{\sqrt{x^2 + 6x}},
   \]
   it then follows that
   \[
   g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(2)} = \frac{4}{5}.
   \]

14. \( f(x) = \sqrt{x^2 + 6x} \) for \( x \leq -6 \), \( b = 4 \)
   **Solution** To determine \( g(4) \), we solve \( f(x) = \sqrt{x^2 + 6x} = 4 \) for \( x \). This yields:
   \[
   \begin{align*}
   x^2 + 6x &= 16 \\
   x^2 + 6x - 16 &= 0 \\
   (x + 8)(x - 2) &= 0
   \end{align*}
   \]
   or \( x = -8, 2 \). Because the domain of \( f \) has been restricted to \( x \leq -6 \), we have \( g(4) = -8 \). With
   \[
   f'(x) = \frac{x + 3}{\sqrt{x^2 + 6x}},
   \]
   it then follows that
   \[
   g'(4) = \frac{1}{f'(g(4))} = \frac{1}{f'(-8)} = -\frac{4}{5}.
   \]

15. \( f(x) = \frac{1}{x + 1} \), \( b = \frac{1}{4} \)
   **Solution** \( f(3) = 1/4 \), so \( g(1/4) = 3 \). \( f'(x) = -\frac{1}{(x+1)^2} \) so \( f'(g(1/4)) = f'(3) = -\frac{1}{(3+1)^2} = -1/16 \). Thus, \( g'(1/4) = -16 \).

16. \( f(x) = e^x \), \( b = e \)
   **Solution** \( f(1) = e \) so \( g(e) = 1 \). \( f'(x) = e^x \) so \( f'(g(e)) = f'(1) = e \). Thus, \( g'(x) = 1/e \).

17. Let \( f(x) = x^n \) and \( g(x) = x^{1/n} \). Compute \( g'(x) \) using Theorem 1 and check your answer using the Power Rule.
   **Solution** Note that \( g(x) = f^{-1}(x) \). Therefore,
   \[
   g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(g(x))^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n(x^{1-1/n})} = \frac{x^{1/n-1}}{n} = \frac{1}{n}(x^{1/n-1})
   \]
   which agrees with the Power Rule.

18. Show that \( f(x) = \frac{1}{1+x} \) and \( g(x) = \frac{1-x}{x} \) are inverses. Then compute \( g'(x) \) directly and verify that \( g'(x) = 1/f'(g(x)) \).
   **Solution** Let \( f(x) = \frac{1}{1+x} \) and \( g(x) = \frac{1-x}{x} \). Then
   \[
   f(g(x)) = \frac{1}{1 + \frac{1-x}{x}} = \frac{x}{x + 1 - x} = x,
   \]
and
\[ g(f(x)) = \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}} = \frac{1 + x - 1}{1} = x; \]

consequently, \( f \) and \( g \) are inverses. Rewriting \( g(x) = x^{-1} - 1 \), we see that \( g'(x) = -x^{-2} \). Moreover, \( f'(x) = -(1 + x)^{-2} \), so
\[ f'(g(x)) = -\left( \frac{1}{x} \right)^{-2} = -(x^{-1})^{-2} = -x^2, \]

and
\[ \frac{1}{f'(g(x))} = -x^{-2} = g'(x). \]

In Exercises 19–22, compute the derivative at the point indicated without using a calculator.

19. \( y = \sin^{-1} x, \ x = \frac{3}{5} \)

**Solution** Let \( y = \sin^{-1} x \). Then \( y' = \frac{1}{\sqrt{1-x^2}} \) and
\[ y' \left( \frac{3}{5} \right) = \frac{1}{\sqrt{1 - (\frac{3}{5})^2}} = \frac{1}{\sqrt{1 - \frac{9}{25}}} = \frac{5}{4}. \]

20. \( y = \tan^{-1} x, \ x = \frac{1}{2} \)

**Solution** Let \( y = \tan^{-1} x \). Then \( y' = \frac{1}{x^2+1} \) and
\[ y' \left( \frac{1}{2} \right) = \frac{1}{\frac{1}{4} + 1} = \frac{4}{5}. \]

21. \( y = \sec^{-1} x, \ x = 4 \)

**Solution** Let \( y = \sec^{-1} x \). Then \( y' = \frac{1}{x|\sqrt{x^2-1}|} \) and
\[ y'(4) = \frac{1}{4\sqrt{15}}. \]

22. \( y = \arccos(4x), \ x = \frac{1}{2} \)

**Solution** Let \( y = \cos^{-1}(4x) \). Then \( y' = \frac{-4}{\sqrt{1-16x^2}} \) and
\[ y' \left( \frac{1}{2} \right) = \frac{-4}{\sqrt{1 - \frac{16}{25}}} = \frac{-4}{\frac{3}{5}} = -\frac{20}{3}. \]

In Exercises 23–36, find the derivative.

23. \( y = \sin^{-1}(7x) \)

**Solution** \( \frac{d}{dx} \sin^{-1}(7x) = \frac{1}{\sqrt{1-(7x)^2}} \)
\[ \frac{d}{dx} 7x = \frac{7}{\sqrt{1-(7x)^2}}. \]

24. \( y = \arctan \left( \frac{x}{3} \right) \)

**Solution** \( \frac{d}{dx} \tan^{-1} \left( \frac{x}{3} \right) = \frac{1}{\left(\frac{x}{3}\right)^2+1} \)
\[ \frac{d}{dx} \left( \frac{x}{3} \right) = \frac{1}{3} \left( \frac{x}{3} \right)^2 + 1 = \frac{1}{\left(\frac{x}{3}\right)^2+3}. \]

25. \( y = \cos^{-1}(x^2) \)

**Solution** \( \frac{d}{dx} \cos^{-1}(x^2) = \frac{-1}{\sqrt{1-x^4}} \)
\[ \frac{d}{dx} x^2 = \frac{-2x}{\sqrt{1-x^4}}. \]

26. \( y = \sec^{-1}(t+1) \)

**Solution** \( \frac{d}{dt} \sec^{-1}(t+1) = \frac{1}{|t+1|\sqrt{(t+1)^2-1}} = \frac{1}{|t+1|\sqrt{t^2+2t}} \)

27. \( y = x \tan^{-1} x \)
SOLUTION \( \frac{d}{dx} \arctan x = x \left( \frac{1}{1 + x^2} \right) + \arctan x. \)

28. \( y = e^{\cos^{-1} x} \)

SOLUTION \( \frac{d}{dx} e^{\cos^{-1} x} = e^{\cos^{-1} x} \cdot \frac{d}{dx} \cos^{-1} x = -e^{\cos^{-1} x} \cdot \frac{1}{\sqrt{1 - x^2}}. \)

29. \( y = \arcsin(e^x) \)

SOLUTION \( \frac{d}{dx} \arcsin(e^x) = \frac{1}{\sqrt{1 - e^{2x}}} \cdot \frac{d}{dx} e^x = \frac{e^x}{\sqrt{1 - e^{2x}}}. \)

30. \( y = \csc^{-1}(x^{-1}) \)

SOLUTION \( \frac{d}{dx} \csc^{-1}(x^{-1}) = \frac{-1}{|x|\sqrt{1/x^2 - 1}} \cdot \frac{1}{x^2} = \frac{1}{|x|\sqrt{1/x^2 - 1}} = \frac{1}{\sqrt{1 - x^2}}. \)

31. \( y = \sqrt{1-t^2} + \sin^{-1} t \)

SOLUTION \( \frac{d}{dt} \left( \sqrt{1-t^2} + \sin^{-1} t \right) = \frac{1}{2} (1-t^2)^{-1/2} (-2t) + \frac{1}{\sqrt{1-t^2}} = -\frac{t}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} = \frac{1-t}{\sqrt{1-t^2}}. \)

32. \( y = \tan^{-1} \left( \frac{1+t}{1-t} \right) \)

SOLUTION \( \frac{d}{dx} \tan^{-1} \left( \frac{1+t}{1-t} \right) = \frac{1}{\left( \frac{1+t}{1-t} \right)^2 + 1} \cdot \frac{(1-t) - (1+t)(-1)}{(1-t)^2} = \frac{2}{(1+t)^2 + (1-t)^2} = \frac{1}{t^2 + 1}. \)

33. \( y = (\tan^{-1} x)^3 \)

SOLUTION \( \frac{d}{dx} \left( (\tan^{-1} x)^3 \right) = 3(\tan^{-1} x)^2 \cdot \frac{d}{dx} \tan^{-1} x = \frac{3(\tan^{-1} x)^2}{x^2 + 1}. \)

34. \( y = \frac{\cos^{-1} x}{\sin^{-1} x} \)

SOLUTION \( \frac{d}{dx} \left( \frac{\cos^{-1} x}{\sin^{-1} x} \right) = \frac{\sin^{-1} x \left( \frac{-1}{\sqrt{1-x^2}} \right) - \cos^{-1} x \left( \frac{1}{\sqrt{1-x^2}} \right)}{(\sin^{-1} x)^2} = \frac{-\pi}{2\sqrt{1-x^2}(\sin^{-1} x)^2}. \)

35. \( y = \cos^{-1} t^{-1} - \sec^{-1} t \)

SOLUTION \( \frac{d}{dx} (\cos^{-1} t^{-1} - \sec^{-1} t) = \frac{-1}{\sqrt{1-(1/t)^2}} \cdot \frac{-1}{t^2} - \frac{1}{|t|\sqrt{t^2 - 1}} = \frac{1}{\sqrt{t^4 - t^2}} - \frac{1}{|t|\sqrt{t^2 - 1}} = \frac{1}{|t|\sqrt{t^2 - 1}} = 0. \)

Alternately, let \( t = \sec \theta \). Then \( t^{-1} = \cos \theta \) and \( \cos^{-1} t^{-1} - \sec^{-1} t = \theta - \theta = 0 \). Consequently,

\( \frac{d}{dx} (\cos^{-1} t^{-1} - \sec^{-1} t) = 0. \)

36. \( y = \cos^{-1} (x + \sin^{-1} x) \)

SOLUTION \( \frac{d}{dx} \cos^{-1} (x + \sin^{-1} x) = \frac{-1}{\sqrt{1-(x + \sin^{-1} x)^2}} \left( 1 + \frac{1}{\sqrt{1-x^2}} \right). \)

37. Use Figure 1 to prove that \( (\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}. \)

\[
\begin{align*}
\text{FIGURE 1 Right triangle with } \theta = \cos^{-1} x.
\end{align*}
\]

SOLUTION Let \( \theta = \cos^{-1} x \). Then \( \cos \theta = x \) and

\[
-\sin \theta \frac{d \theta}{dx} = 1 \quad \text{or} \quad \frac{d \theta}{dx} = -\frac{1}{\sin \theta} = -\frac{1}{\sin(\cos^{-1} x)}.
\]

From Figure 1, we see that \( \sin(\cos^{-1} x) = \sin \theta = \sqrt{1-x^2} \); hence,

\[
\frac{d}{dx} \cos^{-1} x = \frac{1}{-\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.
\]
38. Show that \((\tan^{-1} x)' = \cos^2(\tan^{-1} x)\) and then use Figure 2 to prove that \((\tan^{-1} x)' = (x^2 + 1)^{-1} \). \[ \text{FIGURE 2 Right triangle with } \theta = \tan^{-1} x. \]

**SOLUTION** Let \(\theta = \tan^{-1} x\). Then \(x = \tan \theta\) and

\[
1 = \sec^2 \theta \frac{d\theta}{dx} \quad \text{or} \quad \frac{d\theta}{dx} = \frac{1}{\sec^2 \theta} = \cos^2(\tan^{-1} x).
\]

From Figure 2, \(\cos \theta = \frac{1}{\sqrt{1 + x^2}}\), thus \(\cos^2 \theta = \frac{1}{1 + x^2}\) and

\[
\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}.
\]

39. Let \(\theta = \sec^{-1} x\). Show that \(\tan \theta = \sqrt{x^2 - 1}\) if \(x \geq 1\) and that \(\tan \theta = -\sqrt{x^2 - 1}\) if \(x \leq -1\). *Hint:* \(\tan \theta \geq 0\) on \((0, \frac{\pi}{2})\) and \(\tan \theta \leq 0\) on \((\frac{\pi}{2}, \pi)\).

**SOLUTION** In general, \(1 + \tan^2 \theta = \sec^2 \theta\), so \(\tan \theta = \pm \sqrt{\sec^2 \theta - 1}\). With \(\theta = \sec^{-1} x\), it follows that \(\sec \theta = x\), so \(\tan \theta = \pm \sqrt{x^2 - 1}\). Finally, if \(x \geq 1\) then \(\theta = \sec^{-1} x \in [0, \pi/2]\) so \(\tan \theta\) is positive; on the other hand, if \(x \leq 1\) then \(\theta = \sec^{-1} x \in (-\pi/2, 0]\) so \(\tan \theta\) is negative.

40. Use Exercise 39 to verify the formula

\[
(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2 - 1}}
\]

**SOLUTION** Let \(\theta = \sec^{-1} x\). Then \(\sec \theta = x\) and

\[
\sec \theta \tan \theta \frac{d\theta}{dx} = 1 \quad \text{or} \quad \frac{d\theta}{dx} = \frac{1}{\sec \theta \tan \theta} = \frac{1}{x \tan(\sec^{-1} x)}.
\]

By Exercise 39, \(\tan(\sec^{-1} x) = \sqrt{x^2 - 1}\) for \(x > 1\) and \(\tan(\sec^{-1} x) = -\sqrt{x^2 - 1}\) for \(x < -1\). Hence,

\[
\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}.
\]

### Further Insights and Challenges

41. Let \(g(x)\) be the inverse of \(f(x)\). Show that if \(f'(x) = f(x)\), then \(g'(x) = x^{-1}\). We will apply this in the next section to show that the inverse of \(f(x) = e^x\) (the natural logarithm) has the derivative \(f'(x) = x^{-1}\).

**SOLUTION**

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f(f^{-1}(x))} = \frac{1}{x}.
\]

### 3.9 Derivatives of General Exponential and Logarithmic Functions

#### Preliminary Questions

1. What is the slope of the tangent line to \(y = 4^x\) at \(x = 0\)?

**SOLUTION** The slope of the tangent line to \(y = 4^x\) at \(x = 0\) is

\[
\frac{d}{dx}(4^x) \bigg|_{x=0} = 4^x \ln 4 \bigg|_{x=0} = \ln 4.
\]

2. What is the rate of change of \(y = \ln x\) at \(x = 10\)?

**SOLUTION** The rate of change of \(y = \ln x\) at \(x = 10\) is

\[
\frac{d}{dx}(\ln x) \bigg|_{x=10} = \frac{1}{x} \bigg|_{x=10} = \frac{1}{10}.
\]