54. Use the Product Rule twice to find a formula for \((fg)''\) in terms of \(f\) and \(g\) and their first and second derivatives.

**SOLUTION** Let \( h = fg \). Then \( h' = fg' + gf' = f'g + fg' \) and
\[
h'' = f''g + 2f'g' + g'' + 2gf' + fg''.
\]

55. Use the Product Rule to find a formula for \((fg)^n\) and compare your result with the expansion of \((a + b)^3\). Then try to guess the general formula for \((fg)^n\).

**SOLUTION** Continuing from Exercise 54, we have
\[
h''' = f''''g + 2f''g' + g'' + 2gf' + fg''.
\]
The binomial theorem gives
\[
(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3
\]
and more generally
\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,
\]
where the binomial coefficients are given by
\[
\binom{n}{k} = \frac{k(k-1) \cdots (k-n+1)}{n!}.
\]
Accordingly, the general formula for \((fg)^n\) is given by
\[
(fg)^n = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},
\]
where \( p^{(k)} \) is the \( k \)th derivative of \( p \) (or \( p \) itself when \( k = 0 \)).

56. Compute
\[
\Delta f(x) = \lim_{h\to0} \frac{f(x + h) + f(x - h) - 2f(x)}{h^2}
\]
for the following functions:

(a) \( f(x) = x \)  \hspace{1cm} (b) \( f(x) = x^2 \)  \hspace{1cm} (c) \( f(x) = x^3 \)

Based on these examples, what do you think the limit \( \Delta f \) represents?

**SOLUTION** For \( f(x) = x \), we have
\[
f(x + h) + f(x - h) - 2f(x) = (x + h) + (x - h) - 2x = 0.
\]
Hence, \( \Delta(x) = 0 \). For \( f(x) = x^2 \),
\[
f(x + h) + f(x - h) - 2f(x) = (x + h)^2 + (x - h)^2 - 2x^2
\]
\[
= x^2 + 2xh + h^2 + x^2 - 2xh + h^2 - 2x^2 = 2h^2.
\]
so \( \Delta(x^2) = 2 \). Working in a similar fashion, we find \( \Delta(x^3) = 6x \). One can prove that for twice differentiable functions, \( \Delta f = f'' \). It is an interesting fact of more advanced mathematics that there are functions \( f \) for which \( \Delta f \) exists at all points, but the function is not differentiable.

### 3.6 Trigonometric Functions

**Preliminary Questions**

1. Determine the sign (+ or −) that yields the correct formula for the following:

(a) \( \frac{d}{dx} (\sin x + \cos x) = \pm \sin x \pm \cos x \)

(b) \( \frac{d}{dx} \sec x = \pm \sec x \tan x \)

(c) \( \frac{d}{dx} \cot x = \pm \csc^2 x \)
SOLUTION The correct formulas are

(a) \( \frac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x \)

(b) \( \frac{d}{dx}\sec x = \sec x \tan x \)

(c) \( \frac{d}{dx}\cot x = -\csc^2 x \)

2. Which of the following functions can be differentiated using the rules we have covered so far?

(a) \( y = 3 \cos x \cot x \)

(b) \( y = \cos(x^2) \)

(c) \( y = e^x \sin x \)

SOLUTION

(a) \( 3 \cos x \cot x \) is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

(b) \( \cos(x^2) \) is a composition of the functions \( \cos x \) and \( x^2 \). We have not yet discussed how to differentiate composite functions.

(c) \( x^2 \cos x \) is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

3. Compute \( \frac{d}{dx}(\sin^2 x + \cos^2 x) \) without using the derivative formulas for \( \sin x \) and \( \cos x \).

SOLUTION Recall that \( \sin^2 x + \cos^2 x = 1 \) for all \( x \). Thus,

\[
\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}1 = 0.
\]

4. How is the addition formula used in deriving the formula \( \sin(x + h) \)?

SOLUTION The difference quotient for the function \( \sin x \) involves the expression \( \sin(x + h) \). The addition formula for the sine function is used to expand this expression as \( \sin(x + h) = \sin x \cos h + \sin h \cos x \).

**Exercises**

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. \( y = \sin x, \quad x = \frac{\pi}{4} \)

SOLUTION Let \( f(x) = \sin x \). Then \( f'(x) = \cos x \) and the equation of the tangent line is

\[
y = f'(\frac{\pi}{4})\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}(1 - \frac{\pi}{4}).
\]

2. \( y = \cos x, \quad x = \frac{\pi}{4} \)

SOLUTION Let \( f(x) = \cos x \). Then \( f'(x) = -\sin x \) and the equation of the tangent line is

\[
y = f'(\frac{\pi}{4})\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2} + \frac{\pi}{4}.
\]

3. \( y = \tan x, \quad x = \frac{\pi}{4} \)

SOLUTION Let \( f(x) = \tan x \). Then \( f'(x) = \sec^2 x \) and the equation of the tangent line is

\[
y = f'(\frac{\pi}{4})\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\left(x - \frac{\pi}{4}\right) + 1 = 2x + 1 - \frac{\pi}{2}.
\]

4. \( y = \sec x, \quad x = \frac{\pi}{6} \)

SOLUTION Let \( f(x) = \sec x \). Then \( f'(x) = \sec x \tan x \) and the equation of the tangent line is

\[
y = f'(\frac{\pi}{6})\left(x - \frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right) = \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}} = \frac{2}{3}x + \frac{2\sqrt{3}}{3} + \frac{\pi}{9}.
\]

In Exercises 5–24, compute the derivative.

5. \( f(x) = \sin x \cos x \)

SOLUTION Let \( f(x) = \sin x \cos x \). Then

\[
f'(x) = \sin x(-\sin x) + \cos x(\cos x) = -\sin^2 x + \cos^2 x.
\]
6. \( f(x) = x^2 \cos x \)

**SOLUTION** Let \( f(x) = x^2 \cos x \). Then

\[
\frac{d}{dx} (x^2 \cos x) = x^2 (-\sin x) + (\cos x) (2x) = 2x \cos x - x^2 \sin x.
\]

7. \( f(x) = \sin^2 x \)

**SOLUTION** Let \( f(x) = \sin^2 x = \sin x \sin x \). Then

\[
\frac{d}{dx} (\sin x \cos x) = \sin x (\cos x) + \sin x (\cos x) = 2 \sin x \cos x.
\]

8. \( f(x) = 9 \sec x + 12 \cot x \)

**SOLUTION** Let \( f(x) = 9 \sec x + 12 \cot x \). Then \( f'(x) = 9 \sec x \tan x - 12 \csc^2 x \).

9. \( H(t) = \sin t \sec^2 t \)

**SOLUTION** Let \( H(t) = \sin t \sec^2 t \). Then

\[
H'(t) = \sin t \frac{d}{dt} (\sec t \cdot \sec t) + \sec^2 t (\cos t)
= \sin t (\sec t \tan t + \sec t \tan t) + \sec t 
= 2 \sin t \sec^2 t \tan t + \sec t.
\]

10. \( h(t) = 9 \csc t + t \cot t \)

**SOLUTION** Let \( h(t) = 9 \csc t + t \cot t \). Then

\[
h'(t) = 9 (-\csc t \cot t) + (t (-\csc^2 t)) + \cot t = \cot t - 9 \csc t \cot t - t \csc^2 t.
\]

11. \( f(\theta) = \tan \theta \sec \theta \)

**SOLUTION** Let \( f(\theta) = \tan \theta \sec \theta \). Then

\[
f'(\theta) = \tan \theta \sec \theta \tan \theta + \sec \theta \sec^2 \theta = \sec \theta \tan^2 \theta + \sec^3 \theta = \left( \tan^2 \theta + \sec^2 \theta \right) \sec \theta.
\]

12. \( k(\theta) = \theta^2 \sin^2 \theta \)

**SOLUTION** Let \( k(\theta) = \theta^2 \sin^2 \theta \). Then

\[
k'(\theta) = \theta^2 (2 \sin \theta \cos \theta) + 2 \theta \sin^2 \theta = 2 \theta^2 \sin \theta \cos \theta + 2 \theta \sin^2 \theta.
\]

Here we used the result from Exercise 7.

13. \( f(x) = (2x^4 - 4x^{-1}) \sec x \)

**SOLUTION** Let \( f(x) = (2x^4 - 4x^{-1}) \sec x \). Then

\[
f'(x) = (2x^4 - 4x^{-1}) \sec x \tan x + \sec x (8x^3 + 4x^{-2}).
\]

14. \( f(z) = z \tan z \)

**SOLUTION** Let \( f(z) = z \tan z \). Then \( f'(z) = z(\sec^2 z) + \tan z \).

15. \( y = \frac{\sec \theta}{\theta} \)

**SOLUTION** Let \( y = \frac{\sec \theta}{\theta} \). Then

\[
y' = \frac{\theta \sec \theta \tan \theta - \sec \theta}{\theta^2}.
\]

16. \( G(z) = \frac{1}{\tan z - \cot z} \)

**SOLUTION** Let \( G(z) = \frac{1}{\tan z - \cot z} \). Then

\[
G'(z) = \frac{(\tan z - \cot z)(0) - 1(\sec^2 z + \csc^2 z)}{(\tan z - \cot z)^2} = -\frac{\sec^2 z + \csc^2 z}{(\tan z - \cot z)^2}.
\]

17. \( R(y) = \frac{3 \cos y - 4}{\sin y} \)
24. Let \( R(y) = \frac{3 \cos y - 4}{\sin y} \). Then
\[
R'(y) = \frac{\sin y(-3 \sin y) - (3 \cos y - 4)(\cos y)}{\sin^2 y} = \frac{4 \cos y - 3(\sin^2 y + \cos^2 y)}{\sin^2 y} = \frac{4 \cos y - 3}{\sin^2 y}.
\]

18. \( f(x) = \frac{x}{\sin x + 2} \)

SOLUTION Let \( f(x) = \frac{x}{2 + \sin x} \). Then
\[
f'(x) = \frac{(2 + \sin x)(1) - x \cos x}{(2 + \sin x)^2} = \frac{2 + \sin x - x \cos x}{(2 + \sin x)^2}.
\]

19. \( f(x) = \frac{1 + \tan x}{1 - \tan x} \)

SOLUTION Let \( f(x) = \frac{1 + \tan x}{1 - \tan x} \). Then
\[
f'(x) = \frac{(1 - \tan x) \sec^2 x - (1 + \tan x)(-\sec^2 x)}{(1 - \tan x)^2} = \frac{2 \sec^2 x}{(1 - \tan x)^2}.
\]

20. \( f(\theta) = \theta \tan \theta \sec \theta \)

SOLUTION Let \( f(\theta) = \theta \tan \theta \sec \theta \). Then
\[
f'(\theta) = \theta \frac{d}{d\theta}(\tan \theta \sec \theta) + \tan \theta \sec \theta
\]
\[
= \theta(\tan \theta \sec \theta \tan \theta + \sec \theta \sec^2 \theta) + \tan \theta \sec \theta
\]
\[
= \theta \tan^2 \theta \sec \theta + \theta \sec^3 \theta + \tan \theta \sec \theta.
\]

21. \( f(x) = e^x \sin x \)

SOLUTION Let \( f(x) = e^x \sin x \). Then \( f'(x) = e^x \cos x + \sin x e^x = e^x(\cos x + \sin x) \).

22. \( h(t) = e^{t^3} \tan(t) \)

SOLUTION Let \( h(t) = e^{t^3} \tan(t) \). Then \( h'(t) = e^{t^3} (\sec^2 t \cot(t)) + \tan(t) e^{t^3} = e^{t^3} \sec(t)(1 - \cot(t)) \).

23. \( f(\theta) = e^\theta(5 \sin \theta - 4 \tan \theta) \)

SOLUTION Let \( f(\theta) = e^\theta(5 \sin \theta - 4 \tan \theta) \). Then
\[
f'(\theta) = e^\theta(5 \cos \theta - 4 \sec^2 \theta) + e^\theta(5 \sin \theta - 4 \tan \theta)
\]
\[
= e^\theta(5 \sin \theta + 5 \cos \theta - 4 \tan \theta - 4 \sec^2 \theta).
\]

24. \( f(x) = xe^x \cos x \)

SOLUTION Let \( f(x) = xe^x \cos x \). Then
\[
f'(x) = x \frac{d}{dx}(e^x \cos x) + e^x \cos x = x(e^x(-\sin x) + \cos x e^x) + e^x \cos x
\]
\[
= e^x(x \cos x - x \sin x + \cos x).
\]

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. \( y = x^3 + \cos x, \quad x = 0 \)

SOLUTION Let \( f(x) = x^3 + \cos x \). Then \( f'(x) = 3x^2 - \sin x \) and \( f'(0) = 0 \). The tangent line at \( x = 0 \) is
\[
y = f'(0)(x - 0) + f(0) = 0(x) + 1 = 1.
\]

26. \( y = \tan \theta, \quad \theta = \frac{\pi}{6} \)

SOLUTION Let \( f(\theta) = \tan \theta \). Then \( f'(\theta) = \sec^2 \theta \) and \( f'(\frac{\pi}{6}) = \frac{4}{3} \). The tangent line at \( x = \frac{\pi}{6} \) is
\[
y = f'(\frac{\pi}{6}) \left( \theta - \frac{\pi}{6} \right) + f \left( \frac{\pi}{6} \right) = \frac{4}{3} \left( \theta - \frac{\pi}{6} \right) + \frac{\sqrt{3}}{3} = \frac{4}{3} \theta + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.
\]
27. \( y = \sin x + 3 \cos x, \quad x = 0 \)

**SOLUTION**  Let \( f(x) = \sin x + 3 \cos x \). Then \( f'(x) = \cos x - 3 \sin x \) and \( f'(0) = 1 \). The tangent line at \( x = 0 \) is

\[
y = f'(0)(x - 0) + f(0) = x + 3.
\]

28. \( y = \frac{\sin t}{1 + \cos t}, \quad t = \frac{\pi}{3} \)

**SOLUTION**  Let \( f(t) = \frac{\sin t}{1 + \cos t} \). Then

\[
f'(t) = \frac{(1 + \cos t)(\cos t) - \sin t(-\sin t)}{(1 + \cos t)^2} = \frac{1 + \cos t}{(1 + \cos t)^2} = \frac{1}{1 + \cos t},
\]

and

\[
f'\left(\frac{\pi}{3}\right) = \frac{1}{1 + 1/2} = \frac{2}{3}.
\]

The tangent line at \( x = \frac{\pi}{3} \) is

\[
y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{2}{3} \left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{3} = \frac{2}{3}x + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.
\]

29. \( y = 2(\sin \theta + \cos \theta), \quad \theta = \frac{\pi}{4} \)

**SOLUTION**  Let \( f(\theta) = 2(\sin \theta + \cos \theta) \). Then \( f'(\theta) = 2(\cos \theta - \sin \theta) \) and \( f'(\frac{\pi}{4}) = 1 - \sqrt{2} \). The tangent line at \( x = \frac{\pi}{4} \) is

\[
y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = (1 - \sqrt{2}) \left(x - \frac{\pi}{4}\right) + 1 + \sqrt{2}.
\]

30. \( y = \csc x - \cot x, \quad x = \frac{\pi}{4} \)

**SOLUTION**  Let \( f(x) = \csc x - \cot x \). Then

\[
f'(x) = \csc^2 x - \csc x \cot x
\]

and

\[
f'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \cdot 1 = 2 - \sqrt{2}.
\]

Hence the tangent line is

\[
y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = (2 - \sqrt{2}) \left(x - \frac{\pi}{4}\right) + (\sqrt{2} - 1)
\]

\[
= (2 - \sqrt{2}) x + \sqrt{2} - 1 + \frac{\pi}{4} (\sqrt{2} - 2).
\]

31. \( y = e^x \cos x, \quad x = 0 \)

**SOLUTION**  Let \( f(x) = e^x \cos x \). Then

\[
f'(x) = e^x(-\sin x) + e^x \cos x = e^x \cos (x - \sin x),
\]

and \( f'(0) = e^0(\cos 0 - \sin 0) = 1 \). Thus, the equation of the tangent line is

\[
y = f'(0)(x - 0) + f(0) = x + 1.
\]

32. \( y = e^x \cos^2 x, \quad x = \frac{\pi}{4} \)

**SOLUTION**  Let \( f(x) = e^x \cos^2 x \). Then

\[
f'(x) = e^x \frac{d}{dx}(\cos x \cdot \cos x) + e^x \cos^2 x = e^x(\cos x(-\sin x) + \cos x(-\sin x)) + e^x \cos^2 x
\]

\[
= e^x(\cos^2 x - 2 \sin x \cos x),
\]

and

\[
f'\left(\frac{\pi}{4}\right) = e^{\pi/4} \left(\frac{1}{2} - 1\right) = -\frac{1}{2} e^{\pi/4}.
\]

The tangent line at \( x = \frac{\pi}{4} \) is

\[
y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \frac{1}{2} e^{\pi/4} \left(x - \frac{\pi}{4}\right) + \frac{1}{2} e^{\pi/4}.
\]
In Exercises 35–37, use Theorem 1 to verify the formula.

35. \( \frac{d}{dx} \cot x = -\csc^2 x \)

**SOLUTION**

\[
\frac{d}{dx} \cot x = \frac{\cos x}{\sin x}
\]

Using the quotient rule and the derivative formulas, we compute:

\[
\frac{d}{dx} \cot x = \frac{\cos x}{\sin x} - \frac{\cos x \cdot (-\sin x) - \sin x \cdot (\cos x) \cdot \sin x}{\sin^2 x} = -\frac{(\cos^2 x + \sin^2 x)}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.
\]

36. \( \frac{d}{dx} \sec x = \sec x \tan x \)

**SOLUTION**

Since \( \sec x = \frac{1}{\cos x} \), we can apply the quotient rule and the known derivatives to get:

\[
\frac{d}{dx} \sec x = \frac{1}{\cos x} \frac{d}{dx} \cos x = \frac{\sin x}{\cos x} \sec x = \sin x \frac{1}{\cos x} = \tan x \sec x.
\]

37. \( \frac{d}{dx} \csc x = -\csc x \cot x \)

**SOLUTION**

Since \( \csc x = \frac{1}{\sin x} \), we can apply the quotient rule and the two known derivatives to get:

\[
\frac{d}{dx} \csc x = \frac{1}{\sin x} \frac{d}{dx} \sin x = \frac{\sin x}{\cos x} \csc x = -\cos x \frac{1}{\sin x} = -\cot x \csc x.
\]

38. Show that both \( y = \sin x \) and \( y = \cos x \) satisfy \( y'' = -y \).

**SOLUTION**

Let \( y = \sin x \). Then \( y' = \cos x \) and \( y'' = -\sin x = -y \). Similarly, if we let \( y = \cos x \), then \( y' = -\sin x \) and \( y'' = -\cos x = -y \).

In Exercises 39–42, calculate the higher derivative.

39. \( f''(\theta), \quad f(\theta) = \theta \sin \theta \)

**SOLUTION**

Let \( f(\theta) = \theta \sin \theta \). Then

\[
f'(\theta) = \theta \cos \theta + \sin \theta
\]

\[
f''(\theta) = \theta (-\sin \theta) + \cos \theta + \cos \theta = -\theta \sin \theta + 2 \cos \theta.
\]

40. \( \frac{d^2}{dt^2} \cos^2 t \)

**SOLUTION**

\[
\frac{d}{dt} \cos^2 t = \frac{d}{dt} (\cos t \cdot \cos t) = \cos t (-\sin t) + \cos t (-\sin t) = -2 \sin t \cos t
\]

\[
\frac{d^2}{dt^2} \cos^2 t = \frac{d}{dt} (-2 \sin t \cos t) = -2(\sin t (-\sin t) + \cos t \cos t) = -2(\cos^2 t - \sin^2 t).
\]
41. \( y''', \quad y''' = \tan x \)

**SOLUTION** Let \( y = \tan x \). Then \( y' = \sec^2 x \) and by the Chain Rule,

\[
y'' = \frac{d}{dx} \sec^2 x = 2(\sec x)(\sec x \tan x) = 2 \sec^2 x \tan x
\]

\[
y''' = 2 \sec^2 x (\sec^2 x) + (2 \sec^2 x \tan x) \tan x = 2 \sec^4 + 4 \sec^4 x \tan^2 x
\]

42. \( y''', \quad y'' = e^t \sin t \)

**SOLUTION** Let \( y = e^t \sin t \). Then

\[
y' = e^t \cos t + e^t \sin t = e^t(\cos t + \sin t)
\]

\[
y'' = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t \cos t
\]

\[
y''' = 2e^t(-\sin t) + 2e^t \cos t = 2e^t(\cos t - \sin t).
\]

43. Calculate the first five derivatives of \( f(x) = \cos x \). Then determine \( f^{(8)} \) and \( f^{(37)} \).

**SOLUTION** Let \( f(x) = \cos x \).

- Then \( f'(x) = -\sin x \), \( f''(x) = -\cos x \), \( f'''(x) = \sin x \), \( f^{(4)}(x) = \cos x \), and \( f^{(5)}(x) = -\sin x \).

- Accordingly, the successive derivatives of \( f \) cycle among

\[
\{ -\sin x, -\cos x, \sin x, \cos x \}
\]

in that order. Since 8 is a multiple of 4, we have \( f^{(8)}(x) = \cos x \).

- Since 36 is a multiple of 4, we have \( f^{(36)}(x) = \cos x \). Therefore, \( f^{(37)}(x) = -\sin x \).

44. Find \( y^{(157)} \), where \( y = \sin x \).

**SOLUTION** Let \( f(x) = \sin x \). Then the successive derivatives of \( f \) cycle among

\[
\{ \cos x, -\sin x, -\cos x, \sin x \}
\]

in that order. Since 156 is a multiple of 4, we have \( f^{(156)}(x) = \sin x \). Therefore, \( f^{(157)}(x) = \cos x \).

45. Find the values of \( x \) between 0 and \( 2\pi \) where the tangent line to the graph of \( y = \sin x \cos x \) is horizontal.

**SOLUTION** Let \( y = \sin x \cos x \). Then

\[
y' = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x.
\]

When \( y' = 0 \), we have \( \sin x = \pm \cos x \). In the interval \([0, 2\pi]\), this occurs when \( x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \).

46. **[GU]** Plot the graph \( f(\theta) = \sec \theta + \csc \theta \) over \([0, 2\pi]\) and determine the number of solutions to \( f'(\theta) = 0 \) in this interval graphically. Then compute \( f'(\theta) \) and find the solutions.

**SOLUTION** The graph of \( f(\theta) = \sec \theta + \csc \theta \) over \([0, 2\pi]\) is given below. From the graph, it appears that there are two locations where the tangent line would be horizontal; that is, there appear to be two solutions to \( f'(\theta) = 0 \). Now \( f'(\theta) = \sec \theta \tan \theta - \csc \theta \cot \theta \). Setting \( \sec \theta \tan \theta = \csc \theta \cot \theta = 0 \) and then multiplying by \( \sin \theta \tan \theta \) and rearranging terms yields \( \tan^3 \theta = 1 \). Thus, on the interval \([0, 2\pi]\), there are two solutions of \( f'(\theta) = 0 \): \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{5\pi}{4} \).

![Graph of f(\theta) = \sec \theta + \csc \theta over [0, 2\pi]](image)

47. **[GU]** Let \( g(t) = t - \sin t \).

(a) Plot the graph of \( g \) with a graphing utility for \( 0 \leq t \leq 4\pi \).

(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.

(c) For which values of \( t \) in the given range is the tangent line horizontal?

**SOLUTION** Let \( g(t) = t - \sin t \).

(a) Here is a graph of \( g \) over the interval \([0, 4\pi]\).
(b) Since \( g'(t) = 1 - \cos t \geq 0 \) for all \( t \), the slope of the tangent line to \( g \) is always nonnegative.

(e) In the interval \([0, 4\pi)\), the tangent line is horizontal when \( t = 0, 2\pi, 4\pi \).

48. \( \mathcal{C}_\mathcal{R}_5 \) Let \( f(x) = (\sin x)/x \) for \( x \neq 0 \) and \( f(0) = 1 \).

(a) Plot \( f(x) \) on \([-3\pi, 3\pi]\).

(b) Show that \( f'(c) = 0 \) if \( c = \tan c \). Use the numerical root finder on a computer algebra system to find a good approximation to the smallest positive value \( c_0 \) such that \( f'(c_0) = 0 \).

(c) Verify that the horizontal line \( y = f(c_0) \) is tangent to the graph of \( y = f(x) \) at \( x = c_0 \) by plotting them on the same set of axes.

**SOLUTION**

(a) Here is the graph of \( f(x) \) over \([-3\pi, 3\pi]\).

(b) Let \( f(x) = \frac{\sin x}{x} \). Then

\[
f'(x) = \frac{x \cos x - \sin x}{x^2},
\]

To have \( f'(c) = 0 \), it follows that \( c \cos c - \sin c = 0 \), or

\[
\tan c = c.
\]

Using a computer algebra system, we find that the smallest positive value \( c_0 \) such that \( f'(c_0) = 0 \) is \( c_0 = 4.493409 \).

(c) The horizontal line \( y = f(c_0) = 0.217234 \) and the function \( y = f(x) \) are both plotted below. The horizontal line is clearly tangent to the graph of \( f(x) \).

49. **Show that no tangent line to the graph of \( f(x) = \tan x \) has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of \( (\tan x)' \).**

**SOLUTION** Let \( f(x) = \tan x \). Then \( f'(x) = \sec^2 x = \frac{1}{\cos^2 x} \). Note that \( f'(x) = \frac{1}{\cos^2 x} \) has numerator 1; the equation \( f'(x) = 0 \) therefore has no solution. Because the maximum value of \( \cos^2 x \) is 1, the minimum value of \( f'(x) = \frac{1}{\cos^2 x} \) is 1. Hence, the least slope for a tangent line to \( \tan x \) is 1. Here is a graph of \( f' \).
50. The height at time $t$ (in seconds) of a mass, oscillating at the end of a spring, is $s(t) = 300 + 40 \sin t$ cm. Find the velocity and acceleration at $t = \frac{\pi}{4}$ s.

**Solution** Let $s(t) = 300 + 40 \sin t$ be the height. Then the velocity is

$$v(t) = s'(t) = 40 \cos t$$

and the acceleration is

$$a(t) = v'(t) = -40 \sin t.$$  

At $t = \frac{\pi}{4}$, the velocity is $v \left( \frac{\pi}{4} \right) = 20 \text{ cm/sec}$ and the acceleration is $a \left( \frac{\pi}{4} \right) = -20 \sqrt{3} \text{ cm/sec}^2$.

51. The horizontal range $R$ of a projectile launched from ground level at an angle $\theta$ and initial velocity $v_0$ m/s is $R = \left( \frac{v_0^2}{g} \right) \sin \theta \cos \theta$. Calculate $dR/d\theta$. If $\theta = \pi/24$, will the range increase or decrease if the angle is increased slightly? Base your answer on the sign of the derivative.

**Solution** Let $R(\theta) = \left( \frac{v_0^2}{g} \right) \sin \theta \cos \theta$.

$$\frac{dR}{d\theta} = R'(\theta) = \left( \frac{v_0^2}{g} \right) (-\sin^2 \theta + \cos^2 \theta).$$

If $\theta = \pi/24$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, so $|\sin \theta| > |\cos \theta|$, and $dR/d\theta < 0$ (numerically, $dR/d\theta = -0.0264101v_0^2$). At this point, increasing the angle will decrease the range.

52. Show that if $\frac{\pi}{4} < \theta < \pi$, then the distance along the $x$-axis between $\theta$ and the point where the tangent line intersects the $x$-axis is equal to $|\tan \theta|$ (Figure 1).

**Solution** Let $f(x) = \sin x$. Since $f'(x) = \cos x$, this means that the tangent line at $(\theta, \sin \theta)$ is $y = \cos \theta(x - \theta) + \sin \theta$.

When $y = 0$, $x = \theta - \tan \theta$. The distance from $x$ to $\theta$ is then

$$|\theta - (\theta - \tan \theta)| = |\tan \theta|.$$

---

**Further Insights and Challenges**

53. Use the limit definition of the derivative and the addition law for the cosine function to prove that $(\cos x)' = -\sin x$.

**Solution** Let $f(x) = \cos x$. Then

$$f'(x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = \lim_{h \to 0} \cos x \cos h - \sin x \sin h - \cos x$$

$$= \lim_{h \to 0} \left( -\sin x \frac{\sin h}{h} + (\cos x) \frac{\cos h - 1}{h} \right) = (-\sin x) \cdot 1 + (\cos x) \cdot 0 = -\sin x.$$  

54. Use the addition formula for the tangent

$$\tan(x + h) = \frac{\tan x + \tan h}{1 + \tan x \tan h}$$

to compute $(\tan x)'$ directly as a limit of the difference quotients. You will also need to show that $\lim_{h \to 0} \frac{\tan h}{h} = 1$.

**Solution** First note that

$$\lim_{h \to 0} \frac{\tan h}{h} = \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{\cos h} = \lim_{h \to 0} = 1(1) = 1.$$  

Now, using the addition formula for tangent,

$$\frac{\tan(x + h) - \tan x}{h} = \frac{\tan x + \tan h}{1 + \tan x \tan h} - \tan x$$
In other words, \( F = 55 \).

Therefore,

\[
\frac{d}{dx} \tan x = \lim_{h \to 0} \frac{\tan h}{h} \cdot \frac{\sec^2 x}{1 + \tan x \tan h} = \lim_{h \to 0} \frac{\tan h}{h} \cdot \lim_{h \to 0} \frac{\sec^2 x}{1 + \tan x \tan h} = 1(\sec^2 x) = \sec^2 x.
\]

55. Verify the following identity and use it to give another proof of the formula \((\sin x)' = \cos x\).

\[
\sin(x + h) - \sin x = 2 \cos \left( x + \frac{h}{2} \right) \sin \left( \frac{h}{2} \right)
\]

\textit{Hint:} Use the addition formula to prove that \(\sin(a + b) - \sin(a - b) = 2 \cos a \sin b\).

\textbf{SOLUTION} \hspace{1cm} \text{Recall that}

\[
\sin(a + b) = \sin a \cos b + \cos a \sin b
\]

and

\[
\sin(a - b) = \sin a \cos b - \cos a \sin b.
\]

Subtracting the second identity from the first yields

\[
\sin(a + b) - \sin(a - b) = 2 \cos a \sin b.
\]

If we now set \(a = x + \frac{h}{2} \) and \(b = \frac{h}{2} \), then the previous equation becomes

\[
\sin(x + h) - \sin x = 2 \cos \left( x + \frac{h}{2} \right) \sin \left( \frac{h}{2} \right).
\]

Finally, we use the limit definition of the derivative of \(\sin x\) to obtain

\[
\frac{d}{dx} \sin x = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \lim_{h \to 0} \frac{2 \cos \left( x + \frac{h}{2} \right) \sin \left( \frac{h}{2} \right)}{h}
\]

\[
= \lim_{h \to 0} \cos \left( x + \frac{h}{2} \right) \cdot \lim_{h \to 0} \frac{\sin \left( \frac{h}{2} \right)}{\frac{h}{2}} = \cos x \cdot 1 = \cos x.
\]

In other words, \( \frac{d}{dx} (\sin x) = \cos x \).

56. \hspace{1cm} \textit{Show that a nonzero polynomial function} \( y = f(x) \) \textit{cannot satisfy the equation} \( y'' = -y \). \textit{Use this to prove that neither} \( \sin x \) \textit{nor} \( \cos x \) \textit{is a polynomial}. \textit{Can you think of another way to reach this conclusion by considering limits as} \( x \to \infty \)?

\textbf{SOLUTION}

- Let \( p \) \textit{be a nonzero polynomial of degree} \( n \) \textit{and assume} that \( p \) \textit{ satisfies the differential equation} \( y'' + y = 0 \). \textit{Then} \( p'' + p = 0 \) \textit{for all} \( x \). \textit{There are exactly three cases.}

  (a) If \( n = 0 \), then \( p \) \textit{is a constant polynomial and thus} \( p'' = 0 \). \textit{Hence} \( 0 = p'' + p = p \) \textit{or} \( p = 0 \) \textit{(i.e.,} \( p \) \textit{is equal to} \( 0 \) \textit{for all} \( x \) \textit{or} \( p \) \textit{is identically} \( 0 \)). \textit{This is a contradiction, since} \( p \) \textit{is a nonzero polynomial}.

  (b) If \( n = 1 \), then \( p \) \textit{is a linear polynomial and thus} \( p'' = 0 \). \textit{Once again, we have} \( 0 = p'' + p = p \) \textit{or} \( p = 0 \), \textit{a contradiction since} \( p \) \textit{is a nonzero polynomial}.

  (c) If \( n \geq 2 \), then \( p \) \textit{is at least a quadratic polynomial and thus} \( p'' \) \textit{is a polynomial of degree} \( n - 2 \geq 0 \). \textit{Thus} \( q = p'' + p \) \textit{is a polynomial of degree} \( n \geq 2 \). \textit{By assumption, however,} \( p'' + p = 0 \). \textit{Thus} \( q = 0 \), \textit{a polynomial of degree} \( 0 \). \textit{This is a contradiction, since the degree of} \( q \) \textit{is} \( n \geq 2 \).

\textit{CONCLUSION:} In all cases, we have reached a contradiction. \textit{Therefore the assumption} that \( p \) \textit{ satisfies the differential equation} \( y'' + y = 0 \) \textit{ is false}. \textit{Accordingly, a nonzero polynomial cannot} \textit{ satisfy the stated differential equation}.

- Let \( y = \sin x \). \textit{Then} \( y' = \cos x \) \textit{ and} \( y'' = -\sin x \). \textit{Therefore,} \( y'' = -y \). \textit{Now, let} \( y = \cos x \). \textit{Then} \( y' = -\sin x \) \textit{ and} \( y'' = -\cos x \). \textit{Therefore,} \( y'' = -y \). \textit{Because} \( \sin x \) \textit{ and} \( \cos x \) \textit{ are nonzero functions that satisfy} \( y'' = -y \), \textit{ it follows that neither} \( \sin x \) \textit{ nor} \( \cos x \) \textit{ is a polynomial}.
57. Let \( f(x) = x \sin x \) and \( g(x) = x \cos x \).

(a) Show that \( f'(x) = g(x) + \sin x \) and \( g'(x) = -f(x) + \cos x \).

(b) Verify that \( f''(x) = -f(x) + 2 \cos x \) and \( g''(x) = -g(x) - 2 \sin x \).

(c) By further experimentation, try to find formulas for all higher derivatives of \( f \) and \( g \). \( \text{Hint:} \) The \( k \)th derivative depends on whether \( k = 4n, 4n + 1, 4n + 2, \) or \( 4n + 3 \).

**SOLUTION**  Let \( f(x) = x \sin x \) and \( g(x) = x \cos x \).

(a) We examine first derivatives: \( f'(x) = x \cos x + (\sin x) \cdot 1 = g(x) + \sin x \) and \( g'(x) = (x)(-\sin x) + (\cos x) \cdot 1 = -f(x) + \cos x \); i.e., \( f'(x) = g(x) + \sin x \) and \( g'(x) = -f(x) + \cos x \).

(b) Now look at second derivatives: \( f''(x) = g'(x) + \cos x = -f(x) + 2 \cos x \) and \( g''(x) = -f'(x) - \sin x = -g(x) - 2 \sin x \); i.e., \( f''(x) = -f(x) + 2 \cos x \) and \( g''(x) = -g(x) - 2 \sin x \).

(c) The third derivatives are \( f'''(x) = -f'(x) - 2 \sin x = -g(x) - 3 \sin x \) and \( g'''(x) = -g'(x) - 2 \cos x = f(x) - 3 \cos x \); i.e., \( f'''(x) = -g(x) - 3 \sin x \) and \( g'''(x) = f(x) - 3 \cos x \).

The fourth derivatives are \( f^{(4)}(x) = -g'(x) - 3 \cos x = f(x) - 4 \cos x \) and \( g^{(4)}(x) = f'(x) + 3 \sin x = g(x) + 4 \sin x \); i.e., \( f^{(4)}(x) = f(x) - 4 \cos x \) and \( g^{(4)}(x) = g(x) + 4 \sin x \).

We can now see the pattern for the derivatives, which are summarized in the following table. Here \( n = 0, 1, 2, \ldots \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( 4n )</th>
<th>( 4n + 1 )</th>
<th>( 4n + 2 )</th>
<th>( 4n + 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f^{(k)}(x) )</td>
<td>( f(x) - k \cos x )</td>
<td>( g(x) + k \sin x )</td>
<td>( -f(x) + k \cos x )</td>
<td>( -g(x) - k \sin x )</td>
</tr>
<tr>
<td>( g^{(k)}(x) )</td>
<td>( g(x) + k \sin x )</td>
<td>( -f(x) + k \cos x )</td>
<td>( -g(x) - k \sin x )</td>
<td>( f(x) - k \cos x )</td>
</tr>
</tbody>
</table>

58. Figure 2 shows the geometry behind the derivative formula \( (\sin \theta)' = \cos \theta \). Segments \( \overline{BA} \) and \( \overline{BD} \) are parallel to the \( x \)- and \( y \)-axes. Let \( \Delta \sin \theta = \sin(\theta + h) - \sin \theta \). Verify the following statements.

(a) \( \Delta \sin \theta = BC \)

(b) \( \angle BDA = \theta \)  \( \text{Hint:} \) \( \overline{OA} \perp \overline{AD} \).

(c) \( BD = (\cos \theta)AD \)

Now explain the following intuitive argument: If \( h \) is small, then \( BC \approx BD \) and \( AD \approx h \), so \( \Delta \sin \theta \approx (\cos \theta)h \) and \( (\sin \theta)' = \cos \theta \).

SOLUTION (a) We note that \( \sin(\theta + h) \) is the \( y \)-coordinate of the point \( C \) and \( \sin \theta \) is the \( y \)-coordinate of the point \( A \), and therefore also of the point \( B \). Now, \( \Delta \sin \theta = \sin(\theta + h) - \sin \theta \) can be interpreted as the difference between the \( y \)-coordinates of the points \( B \) and \( C \); that is, \( \Delta \sin \theta = BC \).

(b) From the figure, we note that \( \angle OAB = \theta \), so \( \angle BAD = \pi - \theta \) and \( \angle BDA = \theta \).

(c) Using part (b), it follows that

\[
\cos \theta = \frac{BD}{AD} \quad \text{or} \quad BD = (\cos \theta)AD.
\]

For \( h \) “small,” \( BC \approx BD \) and \( AD \) is roughly the length of the arc subtended from \( A \) to \( C \); that is, \( AD \approx 1(h) = h \). Thus, using (a) and (c),

\[
\Delta \sin \theta = BC \approx BD = (\cos \theta)AD \approx (\cos \theta)h.
\]

In the limit as \( h \to 0 \),

\[
\frac{\Delta \sin \theta}{h} \to (\sin \theta)',
\]

so \( (\sin \theta)' = \cos \theta \).