67. According to Eq. (6) in Section 3.2, \( \frac{d}{dx} b^x = m(b) b^x \). Use the Product Rule to show that \( m(ab) = m(a) + m(b) \).

**SOLUTION**

\[
m(ab)(ab)^x = \frac{d}{dx} (ab)^x = \frac{d}{dx} (a^x b^x) = a^x \frac{d}{dx} b^x + b^x \frac{d}{dx} a^x = m(b) a^x b^x + m(a) a^x b^x = (m(a) + m(b))(ab)^x.
\]

Thus, \( m(ab) = m(a) + m(b) \).

### Section 3.4 Rates of Change

**Preliminary Questions**

1. Which units might be used for each rate of change?
   (a) Pressure (in atmospheres) in a water tank with respect to depth
   (b) The rate of a chemical reaction (change in concentration with respect to time with concentration in moles per liter)
   
   **SOLUTION**
   (a) The rate of change of pressure with respect to depth might be measured in atmospheres/meter.
   (b) The reaction rate of a chemical reaction might be measured in moles/(liter-hour).

2. Two trains travel from New Orleans to Memphis in 4 hours. The first train travels at a constant velocity of 90 mph, but the velocity of the second train varies. What was the second train's average velocity during the trip?

   **SOLUTION**
   Since both trains travel the same distance in the same amount of time, they have the same average velocity: 90 mph.

3. Estimate \( f(26) \), assuming that \( f(25) = 43 \), \( f'(25) = 0.75 \).

   **SOLUTION**
   \( f(x) \approx f(25) + f'(25)(x-25) \), so \( f(26) \approx 43 + 0.75(26-25) = 43.75 \).

4. The population \( P(t) \) of Freedonia in 2009 was \( P(2009) = 5 \) million.
   (a) What is the meaning of \( P'(2009) \)?
   (b) Estimate \( P(2010) \) if \( P'(2009) = 0.2 \).

   **SOLUTION**
   (a) Because \( P(t) \) measures the population of Freedonia as a function of time, the derivative \( P'(2009) \) measures the rate of change of the population of Freedonia in the year 2009.
   (b) \( P(2010) \approx P(2009) + P'(2010) \). Thus, if \( P'(2009) = 0.2 \), then \( P(2009) \approx 5.2 \) million.

### Exercises

In Exercises 1–8, find the rate of change.

1. Area of a square with respect to its side \( s \) when \( s = 5 \).

   **SOLUTION**
   Let the area be \( A = f(s) = s^2 \). Then the rate of change of \( A \) with respect to \( s \) is \( \frac{d}{ds}(s^2) = 2s \). When \( s = 5 \), the area changes at a rate of 10 square units per unit increase. (Draw a \( 5 \times 5 \) square on graph paper and trace the area added by increasing each side length by 1, excluding the corner, to see what this means.)

2. Volume of a cube with respect to its side \( s \) when \( s = 5 \).

   **SOLUTION**
   Let the volume be \( V = f(s) = s^3 \). Then the rate of change of \( V \) with respect to \( s \) is \( \frac{d}{ds} s^3 = 3s^2 \). When \( s = 5 \), the volume changes at a rate of \( 3(5^2) = 75 \) cubic units per unit increase.

3. Cube root \( \sqrt[3]{x} \) with respect to \( x \) when \( x = 1, 8, 27 \).

   **SOLUTION**
   Let \( f(x) = \sqrt[3]{x} \). Writing \( f(x) = x^{1/3} \), we see the rate of change of \( f(x) \) with respect to \( x \) is given by \( f'(x) = \frac{1}{3}x^{-2/3} \). The requested rates of change are given in the table that follows:

<table>
<thead>
<tr>
<th>( c )</th>
<th>ROC of ( f(x) ) with respect to ( x ) at ( x = c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( f'(1) = \frac{1}{3}(1)^{1/3} = \frac{1}{3} )</td>
</tr>
<tr>
<td>8</td>
<td>( f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{3}(\frac{1}{8}) = \frac{1}{24} )</td>
</tr>
<tr>
<td>27</td>
<td>( f'(27) = \frac{1}{3}(27^{-2/3}) = \frac{1}{3}(\frac{1}{27}) = \frac{1}{81} )</td>
</tr>
</tbody>
</table>

4. The reciprocal \( 1/x \) with respect to \( x \) when \( x = 1, 2, 3 \).
SOLUTION Let \( f(x) = x^{-1} \). The rate of change of \( f(x) \) with respect to \( x \) is given by \( f'(x) = -x^{-2} \). The requested rates of change are then \(-1\) when \( x = 1 \), \(-\frac{1}{2}\) when \( x = 2 \) and \(-\frac{1}{9}\) when \( x = 3 \).

5. The diameter of a circle with respect to radius.

SOLUTION The relationship between the diameter \( d \) of a circle and its radius \( r \) is \( d = 2r \). The rate of change of the diameter with respect to the radius is then \( d' = 2 \).

6. Surface area \( A \) of a sphere with respect to radius \( r \) \( (A = 4\pi r^2) \).

SOLUTION Because \( A = 4\pi r^2 \), the rate of change of the surface area of a sphere with respect to the radius is \( A' = 8\pi r \).

7. Volume \( V \) of a cylinder with respect to radius if the height is equal to the radius.

SOLUTION The volume of the cylinder is \( V = \pi r^2 h = \pi r^3 \). Thus \( dV/dr = 3\pi r^2 \).

8. Speed of sound \( v \) (in m/s) with respect to air temperature \( T \) (in kelvins), where \( v = 20\sqrt{T} \).

SOLUTION Because, \( v = 20\sqrt{T} = 20T^{1/2} \), the rate of change of the speed of sound with respect to temperature is \( v' = 10T^{-1/2} = \frac{10}{\sqrt{T}} \).

In Exercises 9–11, refer to Figure 1, the graph of distance \( s(t) \) from the origin as a function of time for a car trip.

![Distance graph](image)

**FIGURE 1** Distance from the origin versus time for a car trip.

9. Find the average velocity over each interval.
(a) \([0, 0.5]\)  
(b) \([0.5, 1]\)  
(c) \([1, 1.5]\)  
(d) \([1, 2]\)

SOLUTION
(a) The average velocity over the interval \([0, 0.5]\) is

\[
\frac{50 - 0}{0.5 - 0} = 100 \text{ km/hour.}
\]

(b) The average velocity over the interval \([0.5, 1]\) is

\[
\frac{100 - 50}{1 - 0.5} = 100 \text{ km/hour.}
\]

(c) The average velocity over the interval \([1, 1.5]\) is

\[
\frac{100 - 100}{1.5 - 1} = 0 \text{ km/hour.}
\]

(d) The average velocity over the interval \([1, 2]\) is

\[
\frac{50 - 100}{2 - 1} = -50 \text{ km/hour.}
\]

10. At what time is velocity at a maximum?

SOLUTION The velocity is maximum when the slope of the distance versus time curve is most positive. This appears to happen when \( t = 0.5 \) hours.

11. Match the descriptions (i)–(iii) with the intervals (a)–(c).
(i) Velocity increasing
(ii) Velocity decreasing
(iii) Velocity negative
(a) \([0, 0.5]\)
(b) \([2.5, 3]\)
(c) \([1.5, 2]\)

SOLUTION
(a) (i) : The distance curve is increasing, and is also bending upward, so that distance is increasing at an increasing rate.

(b) (ii) : Over the interval \([2.5, 3]\), the distance curve is flattening, showing that the car is slowing down; that is, the velocity is decreasing.

(c) (iii) : The distance curve is decreasing, so the tangent line has negative slope; this means the velocity is negative.
12. Use the data from Table 1 in Example 1 to calculate the average rate of change of Martian temperature $T$ with respect to time $t$ over the interval from 8:36 AM to 9:34 AM.

**SOLUTION** The time interval from 8:36 AM to 9:34 AM has length 58 minutes, and the change in temperature over this time interval is

$$\Delta T = -42 - (-47.7) = 5.7^\circ C.$$

The average rate of change is then

$$\frac{\Delta T}{\Delta t} = \frac{5.7}{58} \approx 0.0983 \text{ C/min} = 5.897^\circ \text{C/hr}.$$

13. Use Figure 3 from Example 1 to estimate the instantaneous rate of change of Martian temperature with respect to time (in degrees Celsius per hour) at $t = 4$ AM.

**SOLUTION** The segment of the temperature graph around $t = 4$ AM appears to be a straight line passing through roughly (1:36, −70) and (4:48, −75). The instantaneous rate of change of Martian temperature with respect to time at $t = 4$ AM is therefore approximately

$$\frac{dT}{dt} \approx \frac{-75 - (-70)}{3.2} = -1.5625^\circ \text{C/hour}.$$

14. The temperature (in °C) of an object at time $t$ (in minutes) is $T(t) = \frac{3}{8}t^2 - 15t + 180$ for $0 \leq t \leq 20$. At what rate is the object cooling at $t = 10$? (Give correct units.)

**SOLUTION** Given $T(t) = \frac{3}{8}t^2 - 15t + 180$, it follows that

$$T'(t) = \frac{3}{4}t - 15 \quad \text{and} \quad T'(10) = \frac{3}{4}(10) - 15 = -7.5^\circ \text{C/min}.$$

At $t = 10$, the object is cooling at the rate of 7.5°C/min.

15. The velocity (in cm/s) of blood molecules flowing through a capillary of radius 0.008 cm is $v = 6.4 \times 10^{-8} - 0.001r^2$, where $r$ is the distance from the molecule to the center of the capillary. Find the rate of change of velocity with respect to $r$ when $r = 0.004$ cm.

**SOLUTION** The rate of change of the velocity of the blood molecules is $v'(r) = -0.002$. When $r = 0.004$ cm, this rate is $-8 \times 10^{-6}$ 1/s.

16. Figure 2 displays the voltage $V$ across a capacitor as a function of time while the capacitor is being charged. Estimate the rate of change of voltage at $t = 20$ s. Indicate the values in your calculation and include proper units. Does voltage change more quickly or more slowly as time goes on? Explain in terms of tangent lines.

**SOLUTION** The tangent line sketched in the figure below appears to pass through the points (10, 3) and (30, 4). Thus, the rate of change of voltage at $t = 20$ seconds is approximately

$$\frac{4 - 3}{30 - 10} = 0.05 \text{ V/s}.$$  

As we move to the right of the graph, the tangent lines to it grow shallower, indicating that the voltage changes more slowly as time goes on.
17. Use Figure 3 to estimate $dT/dh$ at $h = 30$ and 70, where $T$ is atmospheric temperature (in degrees Celsius) and $h$ is altitude (in kilometers). Where is $dT/dh$ equal to zero?

![Figure 3](image)

**FIGURE 3** Atmospheric temperature versus altitude.

**SOLUTION** At $h = 30$ km, the graph of atmospheric temperature appears to be linear passing through the points $(23, -50)$ and $(40, 0)$. The slope of this segment of the graph is then

\[
\frac{0 - (-50)}{40 - 23} = \frac{50}{17} = 2.94;
\]

so

\[
\frac{dT}{dh} \bigg|_{h=30} \approx 2.94 \text{°C/km}.
\]

At $h = 70$ km, the graph of atmospheric temperature appears to be linear passing through the points $(58, 0)$ and $(88, -100)$. The slope of this segment of the graph is then

\[
\frac{-100 - 0}{88 - 58} = \frac{-100}{30} = -3.33;
\]

so

\[
\frac{dT}{dh} \bigg|_{h=70} \approx -3.33 \text{°C/km}.
\]

$dT/dh = 0$ at those points where the tangent line on the graph is horizontal. This appears to happen over the interval [13, 23], and near the points $h = 50$ and $h = 90$.

18. The earth exerts a gravitational force of $F(r) = (2.99 \times 10^{16})/r^2$ newtons on an object with a mass of 75 kg located $r$ meters from the center of the earth. Find the rate of change of force with respect to distance $r$ at the surface of the earth.

**SOLUTION** The rate of change of force is $F'(r) = -5.98 \times 10^{16}/r^3$. Therefore,

\[
F'(6.77 \times 10^6) = -5.98 \times 10^{16}/(6.77 \times 10^6)^3 = -1.93 \times 10^{-4} \text{ N/m}.
\]

19. Calculate the rate of change of escape velocity $v_{esc} = (2.82 \times 10^7) r^{-1/2}$ m/s with respect to distance $r$ from the center of the earth.

**SOLUTION** The rate that escape velocity changes is $v'_{esc}(r) = -1.41 \times 10^7 r^{-3/2}$.

20. The power delivered by a battery to an apparatus of resistance $R$ (in ohms) is $P = 2.25R/(R + 0.5)^2$ watts. Find the rate of change of power with respect to resistance for $R = 3 \Omega$ and $R = 5 \Omega$.

**SOLUTION**

\[
P'(R) = \frac{(R + 0.5)^2 \cdot 2.25 - 2.25R(2R + 1)}{(R + 0.5)^4}.
\]

Therefore, $P'(3) = -0.1312 W/\Omega$ and $P'(5) = -0.0609 W/\Omega$.

21. The position of a particle moving in a straight line during a 5-s trip is $s(t) = t^2 - t + 10$ cm. Find a time $t$ at which the instantaneous velocity is equal to the average velocity for the entire trip.

**SOLUTION** Let $s(t) = t^2 - t + 10, 0 \leq t \leq 5$, with $s$ in centimeters (cm) and $t$ in seconds (s). The average velocity over the $t$-interval $[0, 5]$ is

\[
\frac{s(5) - s(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ cm/s}.
\]

The (instantaneous) velocity is $v(t) = s'(t) = 2t - 1$. Solving $2t - 1 = 4$ yields $t = \frac{5}{2}$ s, the time at which the instantaneous velocity equals the calculated average velocity.
22. The height (in meters) of a helicopter at time $t$ (in minutes) is $s(t) = 600t - 3t^3$ for $0 \leq t \leq 12$.

(a) Plot $s(t)$ and velocity $v(t)$.

(b) Find the velocity at $t = 8$ and $t = 10$.

(c) Find the maximum height of the helicopter.

**SOLUTION**

(a) With $s(t) = 600t - 3t^3$, it follows that $v(t) = 600 - 9t^2$. Plots of the position and the velocity are shown below.

(b) From part (a), we have $v(t) = 600 - 9t^2$. Thus, $v'(8) = 24$ meters/minute and $v'(10) = -300$ meters/minute.

(c) From the graph in part (a), we see that the helicopter achieves its maximum height when the velocity is zero. Solving $600 - 9t^2 = 0$ for $t$ yields

$$t = \sqrt{\frac{600}{9}} = \frac{10}{3} \sqrt{6} \text{ minutes}.$$  

The maximum height of the helicopter is then

$$s\left(\frac{10}{3} \sqrt{6}\right) = \frac{4000}{3} \sqrt{6} \approx 3266 \text{ meters}.$$

23. A particle moving along a line has position $s(t) = t^4 - 18t^2$ m at time $t$ seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (that is, it has zero velocity)?

**SOLUTION** The particle passes through the origin when $s(t) = t^4 - 18t^2 = t^2(t^2 - 18) = 0$. This happens when $t = 0$ seconds and when $t = 3\sqrt{2} \approx 4.24$ seconds. With $s(t) = t^4 - 18t^2$, it follows that $v(t) = s'(t) = 4t^3 - 36t = 4t(t^2 - 9)$. The particle is therefore instantaneously motionless when $t = 0$ seconds and when $t = 3$ seconds.

24. **GU** Plot the position of the particle in Exercise 23. What is the farthest distance to the left of the origin attained by the particle?

**SOLUTION** The plot of the position of the particle in Exercise 23 is shown below. Positive values of position correspond to distance to the right of the origin and negative values correspond to distance to the left of the origin. The most negative value of $s(t)$ occurs at $t = 3$ and is equal to $s(3) = 3^4 - 18(3)^2 = -81$. Thus, the particle achieves a maximum distance to the left of the origin of 81 meters.

25. A bullet is fired in the air vertically from ground level with an initial velocity 200 m/s. Find the bullet’s maximum velocity and maximum height.

**SOLUTION** We employ Galileo’s formula, $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = 200t - 4.9t^2$, where the time $t$ is in seconds (s) and the height $s$ is in meters (m). The velocity is $v(t) = 200 - 9.8t$. The maximum velocity of 200 m/s occurs at $t = 0$. This is the initial velocity. The bullet reaches its maximum height when $v(t) = 200 - 9.8t = 0$; i.e., when $t \approx 20.41$ s. At this point, the height is 2040.82 m.

26. Find the velocity of an object dropped from a height of 300 m at the moment it hits the ground.

**SOLUTION** We employ Galileo’s formula, $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = 300 - 4.9t^2$, where the time $t$ is in seconds (s) and the height $s$ is in meters (m). When the object hits the ground its height is 0. Solve $s(t) = 300 - 4.9t^2 = 0$ to obtain $t \approx 7.8246$ s. (We discard the negative time, which took place before the object was dropped.) The velocity at impact is $v(7.8246) = -9.8(7.8246) \approx -76.68$ m/s. This signifies that the object is *falling* at 76.68 m/s.

27. A ball tossed in the air vertically from ground level returns to earth 4 s later. Find the initial velocity and maximum height of the ball.

**SOLUTION** Galileo’s formula gives $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = v_0t - 4.9t^2$, where the time $t$ is in seconds (s) and the height $s$ is in meters (m). When the ball hits the ground after 4 seconds its height is 0. Solve $0 = s(4) = 4v_0 - 4.9(4)^2$ to obtain $v_0 = 19.6$ m/s. The ball reaches its maximum height when $s'(t) = 0$, that is, when $19.6 - 9.8t = 0$, or $t = 2$ s. At this time, $t = 2$ s,

$$s(2) = 0 + 19.6(2) - \frac{1}{2}(9.8)(4) = 19.6 \text{ m}.$$
28. Olivia is gazing out a window from the tenth floor of a building when a bucket (dropped by a window washer) passes by. She notes that it hits the ground 1.5 s later. Determine the floor from which the bucket was dropped if each floor is 5 m high and the window is in the middle of the tenth floor. Neglect air friction.

**SOLUTION** Suppose \( H \) is the unknown height from which the bucket fell starting at time \( t = 0 \). The height of the bucket at time \( t \) is \( s(t) = H - 4.9t^2 \). Let \( T \) be the time when the bucket hits the ground (thus \( s(T) = 0 \)). Olivia saw the bucket at time \( T - 1.5 \). The window is located 9.5 floors or 47.5 m above ground. So we have the equations

\[
s(T - 1.5) = H - 4.9(T - 1.5)^2 = 47.5 \quad \text{and} \quad s(T) = H - 4.9T^2 = 0
\]

Subtracting the second equation from the first, we obtain \(-4.9(-3T + 2.25) = 47.5\), so \( T \approx 4 \) s. The second equation gives us \( H = 4.9T^2 = 4.9(4)^2 \approx 78.4 \) m. Since there are 5 m in a floor, the bucket was dropped 78.4/5 \( \approx 15.7 \) floors above the ground. The bucket was dropped from the top of the 15th floor.

29. Show that for an object falling according to Galileo’s formula, the average velocity over any time interval \([t_1, t_2]\) is equal to the average of the instantaneous velocities at \( t_1 \) and \( t_2 \).

**SOLUTION** The simplest way to proceed is to compute both values and show that they are equal. The average velocity over \([t_1, t_2]\) is

\[
\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{s_0 + v_0 t_2 - \frac{1}{2}gt_2^2 - (s_0 + v_0 t_1 - \frac{1}{2}gt_1^2)}{t_2 - t_1} = \frac{v_0(t_2 - t_1) + \frac{1}{2}g(t_2^2 - t_1^2)}{t_2 - t_1}
\]

Whereas the average of the instantaneous velocities at the beginning and end of \([t_1, t_2]\) is

\[
\frac{s'(t_1) + s'(t_2)}{2} = \frac{1}{2} \left( (v_0 - gt_1) + (v_0 - gt_2) \right) = \frac{1}{2} (2v_0) - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1).
\]

The two quantities are the same.

30. An object falls under the influence of gravity near the earth’s surface. Which of the following statements is true? Explain.

(a) Distance traveled increases by equal amounts in equal time intervals.

(b) Velocity increases by equal amounts in equal time intervals.

(c) The derivative of velocity increases with time.

**SOLUTION** For an object falling under the influence of gravity, Galileo’s formula gives \( s(t) = s_0 + v_0 t - \frac{1}{2}gt^2 \).

(a) Since the height of the object varies quadratically with respect to time, it is not true that the object covers equal distance in equal time intervals.

(b) The velocity is \( v(t) = s'(t) = v_0 - gt \). The velocity varies linearly with respect to time. Accordingly, the velocity decreases (becomes more negative) by equal amounts in equal time intervals. Moreover, its speed (the magnitude of velocity) increases by equal amounts in equal time intervals.

(c) Acceleration, the derivative of velocity with respect to time, is given by \( a(t) = v'(t) = -g \). This is a constant; it does not change with time. Hence it is not true that acceleration (the derivative of velocity) increases with time.

31. By Faraday’s Law, if a conducting wire of length \( l \) meters moves at velocity \( v \) m/s perpendicular to a magnetic field of strength \( B \) (in teslas), a voltage of size \( V = -B\ell v \) is induced in the wire. Assume that \( B = 2 \) and \( \ell = 0.5 \).

(a) Calculate \( dV/dv \).

(b) Find the rate of change of \( V \) with respect to time \( t \) if \( v = 4t + 9 \).

**SOLUTION**

(a) Assuming that \( B = 2 \) and \( l = 0.5 \), \( V = -2(0.5)v = -v \). Therefore,

\[
\frac{dV}{dv} = -1.
\]

(b) If \( v = 4t + 9 \), then \( V = -2(0.5)(4t + 9) = -(4t + 9) \). Therefore, \( \frac{dV}{dt} = -4 \).

32. The voltage \( V \), current \( I \), and resistance \( R \) in a circuit are related by Ohm’s Law: \( V = IR \), where the units are volts, amperes, and ohms. Assume that voltage is constant with \( V = 12 \) volts. Calculate (specifying units):

(a) The average rate of change of \( I \) with respect to \( R \) for the interval from \( R = 8 \) to \( R = 8.1 \)

(b) The rate of change of \( I \) with respect to \( R \) when \( R = 8 \)

(c) The rate of change of \( R \) with respect to \( I \) when \( I = 1.5 \)
SOLUTION  Let \( V = IR \) or \( I = V/R = 12/R \) (since we are assuming \( V = 12 \) volts).

(a) The average rate of change is

\[
\frac{\Delta I}{\Delta R} = \frac{I(8.1) - I(8)}{8.1 - 8} = \frac{12 - \frac{12}{0.1}}{0.1} \approx -0.185 \text{ A/}^\circ.
\]

(b) \( dI/dR = -12/R^2 = -12/8^2 = -0.1875 \text{ A}/\Omega. \)

(c) With \( R = 12/1 \), we have \( dR/dI = -12/I^2 = -12/1.5^2 \approx -5.33 \text{ A}/\Omega. \)

33. Ethan finds that with \( h \) hours of tutoring, he is able to answer correctly \( S(h) \) percent of the problems on a math exam. Which would you expect to be larger: \( S'(3) \) or \( S'(30) \)? Explain.

SOLUTION  One possible graph of \( S(h) \) is shown in the figure below on the left. This graph indicates that in the early hours of working with the tutor, Ethan makes rapid progress in learning the material but eventually approaches either the limit of his ability to learn the material or the maximum possible score on the exam. In this scenario, \( S'(3) \) would be larger than \( S'(30) \).

An alternative graph of \( S(h) \) is shown below on the right. Here, in the early hours of working with the tutor little progress is made (perhaps the tutor is assessing how much Ethan already knows, his learning style, his personality, etc.). This is followed by a period of rapid improvement and finally a leveling off as Ethan reaches his maximum score. In this scenario, \( S'(3) \) and \( S'(30) \) might be roughly equal.

![Graphs of S(h) for 3 hours and 30 hours of tutoring]

34. Suppose \( \theta(t) \) measures the angle between a clock’s minute and hour hands. What is \( \theta'(t) \) at 3 o’clock?

SOLUTION  The minute hand makes one full revolution every 60 minutes, so the minute hand moves at a rate of

\[
\frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/min.}
\]

The hour hand makes one-twelfth of a revolution every 60 minutes, so the hour hand moves at a rate of

\[
\frac{\pi}{360} \text{ rad/min.}
\]

At 3 o’clock, the movement of the minute hand works to decrease the angle between the minute and hour hands while the movement of the hour hand works to increase the angle. Therefore, at 3 o’clock,

\[
\theta'(t) = \frac{\pi}{360} - \frac{\pi}{30} = -\frac{11\pi}{360} \text{ rad/min.}
\]

35. To determine drug dosages, doctors estimate a person’s body surface area (BSA) (in meters squared) using the formula \( \text{BSA} = \sqrt{hm}/60 \), where \( h \) is the height in centimeters and \( m \) the mass in kilograms. Calculate the rate of change of BSA with respect to mass for a person of constant height \( h = 180 \). What is this rate at \( m = 70 \) and \( m = 80 \)? Express your result in the correct units. Does BSA increase more rapidly with respect to mass at lower or higher body mass?

SOLUTION  Assuming constant height \( h = 180 \) cm, let \( f(m) = \sqrt{hm}/60 = \sqrt{\frac{180}{10}}m \) be the formula for body surface area in terms of weight. The rate of change of BSA with respect to mass is

\[
f'(m) = \frac{\sqrt{5}}{10} \left( \frac{1}{2}m^{-1/2} \right) = \frac{\sqrt{5}}{20\sqrt{m}}.
\]

If \( m = 70 \) kg, this is

\[
f'(70) = \frac{\sqrt{5}}{20\sqrt{70}} = \frac{\sqrt{14}}{280} \approx 0.0133631 \text{ m}^2/\text{kg}.
\]

If \( m = 80 \) kg,

\[
f'(80) = \frac{\sqrt{5}}{20\sqrt{80}} = \frac{1}{20\sqrt{16}} = \frac{1}{80} \text{ m}^2/\text{kg}.
\]

Because the rate of change of BSA depends on \( 1/\sqrt{m} \), it is clear that BSA increases more rapidly at lower body mass.
36. The atmospheric CO\(_2\) level \(A(t)\) at Mauna Loa, Hawaii at time \(t\) (in parts per million by volume) is recorded by the Scripps Institution of Oceanography. The values for the months January–December 2007 were

\[
382.45, 383.68, 384.23, 386.26, 386.39, 385.87, 384.39, 381.78, 380.73, 380.81, 382.33, 383.69
\]

(a) Assuming that the measurements were made on the first of each month, estimate \(A'(t)\) on the 15th of the months January–November.

(b) In which months did \(A'(t)\) take on its largest and smallest values?

(c) In which month was the CO\(_2\) level most nearly constant?

**SOLUTION**

(a) The rate of change in the atmospheric CO\(_2\) level on the 15th of each month can be estimated using the monthly differences \(A(n) - A(n-1)\) for \(2 \leq n \leq 12\). The estimates we obtain are:

\[
\begin{array}{c|cccccccccc}
    \hline
    P'(t) & 1.23 & 0.55 & 2.03 & -0.52 & -1.48 & -2.61 & -1.05 & 0.08 & 1.52 & 1.36 \\
\end{array}
\]

(b) According to the table in part (a), the maximum rate of change occurs in March and the minimum rate is in July.

(c) According to the table in part (a), the CO\(_2\) level is most nearly constant in September.

37. The tangent lines to the graph of \(f(x) = x^2\) grow steeper as \(x\) increases. At what rate do the slopes of the tangent lines increase?

**SOLUTION** Let \(f(x) = x^2\). The slopes \(s\) of the tangent lines are given by \(s = f'(x) = 2x\). The rate at which these slopes are increasing is \(ds/dx = 2\).

38. Figure 4 shows the height \(y\) of a mass oscillating at the end of a spring, through one cycle of the oscillation. Sketch the graph of velocity as a function of time.

**SOLUTION** The position graph appears to break into four equal-sized components. Over the first quarter of the time interval, the position graph is rising but bending downward, eventually reaching a horizontal tangent. Thus, over the first quarter of the time interval, the velocity is positive but decreasing, eventually reaching 0. Continuing to examine the structure of the position graph produces the following graph of velocity:

In Exercises 39–46, use Eq. (3) to estimate the unit change.

39. Estimate \(\sqrt{2} - \sqrt{1}\) and \(\sqrt{101} - \sqrt{100}\). Compare your estimates with the actual values.

**SOLUTION** Let \(f(x) = \sqrt{x}\). Then \(f'(x) = \frac{1}{2}(x^{-1/2})\). We are using the derivative to estimate the average rate of change. That is,

\[
\frac{\sqrt{x + h} - \sqrt{x}}{h} \approx f'(x),
\]

so that

\[
\sqrt{x + h} - \sqrt{x} \approx hf'(x).
\]

Thus, \(\sqrt{2} - \sqrt{1} \approx 1f'(1) = \frac{1}{2}(1) = \frac{1}{2}\). The actual value, to six decimal places, is 0.414214. Also, \(\sqrt{101} - \sqrt{100} \approx 1f'(100) = \frac{1}{200} = 0.005\). The actual value, to six decimal places, is 0.0498756.
40. Estimate \( f(4) - f(3) \) if \( f'(x) = 2^{-x} \). Then estimate \( f(4) \), assuming that \( f(3) = 12 \).

**SOLUTION** Using the estimate that
\[
\frac{f(x + h) - f(x)}{h} \approx f'(x),
\]
so that \( f(x + h) - f(x) \approx f'(x)h \) with \( x = 3, h = 1 \), we get
\[
\frac{f(4) - f(3)}{1} = 2^{-3}(1) = \frac{1}{8}.
\]
If \( f(3) = 12 \), then \( f(4) \approx 12 \frac{1}{8} = \frac{27}{8} \).

41. Let \( F(s) = 1.1s + 0.05s^2 \) be the stopping distance as in Example 3. Calculate \( F(65) \) and estimate the increase in stopping distance if speed is increased from 65 to 66 mph. Compare your estimate with the actual increase.

**SOLUTION** Let \( F(s) = 1.1s + 0.05s^2 \) be as in Example 3. \( F'(s) = 1.1 + 0.1s \).

- Then \( F(65) = 282.75 \) ft and \( F'(65) = 7.6 \) ft/mph,
- \( F'(65) \approx F(66) - F(65) \) is approximately equal to the change in stopping distance per 1 mph increase in speed when traveling at 65 mph. Increasing speed from 65 to 66 therefore increases stopping distance by approximately 7.6 ft.
- The actual increase in stopping distance when speed increases from 65 mph to 66 mph is \( F(66) - F(65) = 290.4 - 282.75 = 7.65 \) feet, which differs by less than one percent from the estimate found using the derivative.

42. According to Kleiber’s Law, the metabolic rate \( P \) (in kilocalories per day) and body mass \( m \) (in kilograms) of an animal are related by a *three-quarter-power law* \( P = 73.3m^{3/4} \). Estimate the increase in metabolic rate when body mass increases from 60 to 61 kg.

**SOLUTION** Let \( P(m) = 73.3m^{3/4} \) be the function relating body mass \( m \) to metabolic rate \( P \). Then,
\[
P'(m) = \frac{3}{4}(73.3)m^{-1/4} = 54.975m^{-1/4}
\]
\[
P(61) - P(60) \approx P'(60) = 54.975(60^{-1/4}) = 19.7527.
\]
As body mass is increased from 60 to 61 kg, metabolic rate is increased by approximately 19.7527 kcal/day.

43. The dollar cost of producing \( x \) bagels is \( C(x) = 300 + 0.25x - 0.5(x/1000)^3 \). Determine the cost of producing 2000 bagels and estimate the cost of the 2001st bagel. Compare your estimate with the actual cost of the 2001st bagel.

**SOLUTION** Expanding the power of 3 yields
\[
C(x) = 300 + 0.25x - 5 \times 10^{-10}x^3.
\]
This allows us to get the derivative \( C'(x) = 0.25 - 1.5 \times 10^{-9}x^2 \). The cost of producing 2000 bagels is
\[
C(2000) = 300 + 0.25(2000) - 0.5(2000/1000)^3 = 796
\]
\[
C'(2000) = 0.25 - 1.5 \times 10^{-9}(2000^2) = \$0.244.
\]
\( C(2001) = 796.244 \), so the *exact* cost of the 2001st bagel is indistinguishable from the estimated cost. The function is very nearly linear at this point.

44. Suppose the dollar cost of producing \( x \) video cameras is \( C(x) = 500x - 0.003x^2 + 10^{-8}x^3 \).

(a) Estimate the marginal cost at production level \( x = 5000 \) and compare it with the actual cost \( C(5001) - C(5000) \).

(b) Compare the marginal cost at \( x = 5000 \) with the average cost per camera, defined as \( C(x)/x \).

**SOLUTION** Let \( C(x) = 500x - 0.003x^2 + 10^{-8}x^3 \). Then
\[
C'(x) = 500 - 0.006x + (3 \times 10^{-8})x^2.
\]
(a) The cost difference is approximately \( C'(5000) = 470.75 \). The actual cost is \( C(5001) - C(5000) = 470.747 \), which is quite close to the marginal cost computed using the derivative.

(b) The average cost per camera is
\[
\frac{C(5000)}{5000} = \frac{2426250}{5000} = 485.25,
\]
which is slightly higher than the marginal cost.
45. Demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is $D(p) = 900/p$ barrels, where $p$ is the dollar price per barrel. Find the demand when $p =$ $40$. Estimate the decrease in demand if $p$ rises to $41$ and the increase if $p$ declines to $39$.

**Solution** $D(p) = 900p^{-1}$, so $D'(p) = -900p^{-2}$. When the price is $40$ a barrel, the per capita demand is $D(40) = 22.5$ barrels per year. With an increase in price from $40$ to $41$ a barrel, the change in demand $D(41) - D(40)$ is approximately $D'(40) = -900(40^{-2}) = -0.5625$ barrels a year. With a decrease in price from $40$ to $39$ a barrel, the change in demand $D(39) - D(40)$ is approximately $-D'(40) = +0.5625$. An increase in oil prices of a dollar leads to a decrease in demand of 0.5625 barrels a year, and a decrease of a dollar leads to an increase in demand of 0.5625 barrels a year.

46. The reproduction rate $f$ of the fruit fly *Drosophila melanogaster*, grown in bottles in a laboratory, decreases with the number $p$ of flies in the bottle. A researcher has found the number of offspring per female per day to be approximately $f(p) = (34 - 0.612p)p^{-0.658}$.

(a) Calculate $f(15)$ and $f'(15)$.

(b) Estimate the decrease in daily offspring per female when $p$ is increased from 15 to 16. Is this estimate larger or smaller than the actual value $f(16) - f(15)$?

(c) Plot $f(p)$ for $5 \leq p \leq 25$ and verify that $f(p)$ is a decreasing function of $p$. Do you expect $f'(p)$ to be positive or negative? Plot $f'(p)$ and confirm your expectation.

**Solution** Let

$$f(p) = (34 - 0.612p)p^{-0.658} = 34p^{-0.658} - 0.612p^{0.342}.$$

Then

$$f'(p) = -22.372p^{-1.658} - 0.209304p^{-0.658}.$$

(a) $f(15) = 34(15)^{-0.658} - 0.612(15)^{0.342} \approx 4.17767$ offspring per female per day and $f'(15) = -22.372(15)^{-1.658} - 0.209304(15)^{-0.658} \approx -0.28627/2$ offspring per female per day per fly.

(b) $f(16) - f(15) \approx f'(15) \approx -0.28627$. The decrease in daily offspring per female is estimated at 0.28627. $f(16) - f(15) = -0.272424$. The actual decrease in daily offspring per female is 0.272424. The actual decrease in daily offspring per female is less than the estimated decrease. This is because the graph of the function bends towards the $x$ axis.

(c) The function $f(p)$ is plotted below at the left and is clearly a decreasing function of $p$; we therefore expect that $f'(p)$ will be negative. The plot of the derivative shown below at the right confirms our expectation.

![Graphs of $f(p)$ and $f'(p)$](image)

47. According to Stevens' Law in psychology, the perceived magnitude of a stimulus is proportional (approximately) to a power of the actual intensity $I$ of the stimulus. Experiments show that the perceived brightness $B$ of a light satisfies $B = kI^{2/3}$, where $I$ is the light intensity, whereas the perceived heaviness $H$ of a weight $W$ satisfies $H = kW^{3/2}$ ($k$ is a constant that is different in the two cases). Compute $dB/dI$ and $dH/dW$ and state whether they are increasing or decreasing functions. Then explain the following statements:

(a) A one-unit increase in light intensity is felt more strongly when $I$ is small than when $I$ is large.

(b) Adding another pound to a load $W$ is felt more strongly when $W$ is large than when $W$ is small.

**Solution**

(a) $dB/dI = \frac{2k}{3}I^{-1/3} = \frac{2k}{3}I^{1/3}$.

As $I$ increases, $dB/dI$ shrinks, so that the rate of change of perceived intensity decreases as the actual intensity increases. Increased light intensity has a diminished return in perceived intensity. A sketch of $B$ against $I$ is shown: See that the height of the graph increases more slowly as you move to the right.

![Graph of $B$ against $I$](image)

(b) $dH/dW = \frac{3k}{2}W^{1/2}$. As $W$ increases, $dH/dW$ increases as well, so that the rate of change of perceived weight increases as weight increases. A sketch of $H$ against $W$ is shown: See that the graph becomes steeper as you move to the right.
48. Let \( M(t) \) be the mass (in kilograms) of a plant as a function of time (in years). Recent studies by Niklas and Enquist have suggested that a remarkably wide range of plants (from algae and grass to palm trees) obey a three-quarter-power growth law—that is, \( dM/dt = CM^{3/4} \) for some constant \( C \).

(a) If a tree has a growth rate of 6 kg/yr when \( M = 100 \) kg, what is its growth rate when \( M = 125 \) kg?

(b) If \( M = 0.5 \) kg, how much more mass must the plant acquire to double its growth rate?

**SOLUTION**

(a) Suppose a tree has a growth rate \( dM/dt \) of 6 kg/yr when \( M = 100 \) kg, then
\[
6 = C(100^{3/4}) = 10C \sqrt[4]{10},
\]
so that \( C = \frac{3 \sqrt[4]{10}}{50} \).

When \( M = 125 \),
\[
\frac{dM}{dt} = C(125^{3/4}) = \frac{3 \sqrt[4]{10}}{50} \times 25(5^{1/4}) = 7.09306.
\]

(b) The growth rate when \( M = 0.5 \) kg is \( dM/dt = C(0.5^{3/4}) \). To double the rate, we must find \( M \) so that \( dM/dt = CM^{3/4} = 2C(0.5^{3/4}) \). We solve for \( M \).
\[
CM^{3/4} = 2C(0.5^{3/4}) \Rightarrow M^{3/4} = 2(0.5^{3/4}) \Rightarrow M = (2(0.5^{3/4}))^{4/3} = 1.25992.
\]

The plant must acquire the difference \( 1.25992 - 0.5 = 0.75992 \) kg in order to double its growth rate.

Note that a doubling of growth rate requires more than a doubling of mass.

**Further Insights and Challenges**

**Exercises 49–51:** The Lorenz curve \( y = F(r) \) is used by economists to study income distribution in a given country (see Figure 5). By definition, \( F(r) \) is the fraction of the total income that goes to the bottom \( r \)th part of the population, where \( 0 \leq r \leq 1 \). For example, if \( F(0.4) = 0.245 \), then the bottom 40% of households receive 24.5% of the total income. Note that \( F(0) = 0 \) and \( F(1) = 1 \).

**FIGURE 5**

49. Our goal is to find an interpretation for \( F'(r) \). The average income for a group of households is the total income going to the group divided by the number of households in the group. The national average income is \( A = T/N \), where \( N \) is the total number of households and \( T \) is the total income earned by the entire population.

(a) Show that the average income among households in the bottom \( r \)th part is equal to \( (F(r)/r)A \).

(b) Show more generally that the average income of households belonging to an interval \([r, r + \Delta r]\) is equal to
\[
\left( \frac{F(r + \Delta r) - F(r)}{\Delta r} \right) A.
\]
(c) Let \( 0 \leq r \leq 1 \). A household belongs to the 100th percentile if its income is greater than or equal to the income of 100r % of all households. Pass to the limit as \( \Delta r \to 0 \) in (b) to derive the following interpretation: A household in the 100th percentile has income \( F'(r)A \). In particular, a household in the 100th percentile receives more than the national average if \( F'(r) > 1 \) and less if \( F'(r) < 1 \).

(d) For the Lorenz curves \( L_1 \) and \( L_2 \) in Figure 5(B), what percentage of households have above-average income?

SOLUTION

(a) The total income among households in the bottom \( r \)th part is \( F(r)T \) and there are \( rN \) households in this part of the population. Thus, the average income among households in the bottom \( r \)th part is equal to

\[
\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r}A.
\]

(b) Consider the interval \([r, r + \Delta r]\). The total income among households between the bottom \( r \)th part and the bottom \( r + \Delta r \)-th part is \( F(r + \Delta r)T - F(r)T \). Moreover, the number of households covered by this interval is \((r + \Delta r)N - rN = \Delta rN\). Thus, the average income of households belonging to an interval \([r, r + \Delta r]\) is equal to

\[
\frac{F(r + \Delta r)T - F(r)T}{\Delta rN} \approx \frac{F(r + \Delta r) - F(r)}{\Delta r}, \quad \frac{T}{N} = \frac{F(r + \Delta r) - F(r)}{\Delta r}A.
\]

(c) Take the result from part (b) and let \( \Delta r \to 0 \). Because

\[
\lim_{\Delta r \to 0} \frac{F(r + \Delta r) - F(r)}{\Delta r} = F'(r),
\]

we find that a household in the 100r%th percentile has income \( F'(r)A \).

(d) The point \( P \) in Figure 5(B) has an \( r \)-coordinate of 0.6, while the point \( Q \) has an \( r \)-coordinate of roughly 0.75. Thus, on curve \( L_1 \), 40% of households have \( F'(r) > 1 \) and therefore have above-average income. On curve \( L_2 \), roughly 25% of households have above-average income.

50. The following table provides values of \( F(r) \) for Sweden in 2004. Assume that the national average income was \( A = 30,000 \) euros.

<table>
<thead>
<tr>
<th>( r )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(r) )</td>
<td>0</td>
<td>0.01</td>
<td>0.245</td>
<td>0.423</td>
<td>0.642</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) What was the average income in the lowest 40% of households?

(b) Show that the average income of the households belonging to the interval \([0.4, 0.6]\) was 26,700 euros.

(c) Estimate \( F'(0.5) \). Estimate the income of households in the 50th percentile? Was it greater or less than the national average?

SOLUTION

(a) The average income in the lowest 40% of households is \( F'(0.4)A = 0.245(30,000) = 7350 \) euros.

(b) The average income of the households belonging to the interval \([0.4, 0.6]\) is

\[
\frac{F(0.6) - F(0.4)}{0.2}A = \frac{0.423 - 0.245}{0.2}(30,000) = (0.89)(30,000) = 26700
\]

euros.

(c) We estimate

\[
F'(0.5) \approx \frac{F(0.6) - F(0.4)}{0.2} = \frac{0.423 - 0.245}{0.2} = 0.89.
\]

The income of households in the 50th percentile is then \( F'(0.5)A = 0.89(30,000) = 26,700 \) euros, which is less than the national average.

51. Use Exercise 49 (c) to prove:

(a) \( F'(r) \) is an increasing function of \( r \).

(b) Income is distributed equally (all households have the same income) if and only if \( F(r) = r \) for \( 0 \leq r \leq 1 \).

SOLUTION

(a) Recall from Exercise 49 (c) that \( F'(r)A \) is the income of a household in the 100r%-th percentile. Suppose \( 0 \leq r_1 < r_2 \leq 1 \). Because \( r_2 > r_1 \), a household in the 100r_2-th percentile must have income at least as large as a household in the 100r_1-th percentile. Thus, \( F'(r_2)A \geq F'(r_1)A \), or \( F'(r_2) \geq F'(r_1) \). This implies \( F'(r) \) is an increasing function of \( r \).

(b) If \( F(r) = r \) for \( 0 \leq r \leq 1 \), then \( F'(r) = 1 \) and households in all percentiles have income equal to the national average; that is, income is distributed equally. Alternately, if income is distributed equally (all households have the same income), then \( F'(r) = 1 \) for \( 0 \leq r \leq 1 \). Thus, \( F \) must be a linear function in \( r \) with slope 1. Moreover, the condition \( F(0) = 0 \) requires the \( F \) intercept of the line to be 0. Hence, \( F(r) = 1 \cdot r + 0 = r \).
52. CAS Studies of Internet usage show that website popularity is described quite well by Zipf’s Law, according to which the th most popular website receives roughly the fraction \(1/n\) of all visits. Suppose that on a particular day, the th most popular site had approximately \(V(n) = 10^6/n\) visitors for \(n \leq 15,000\).

(a) Verify that the top 50 websites received nearly 45% of the visits. Hint: Let \(T(N)\) denote the sum of \(V(n)\) for \(1 \leq n \leq N\). Use a computer algebra system to compute \(T(50)\) and \(T(15,000)\).

(b) Verify, by numerical experimentation, that when Eq. (3) is used to estimate \(V(n + 1) - V(n)\), the error in the estimate decreases as \(n\) grows larger. Find (again, by experimentation) an \(N\) such that the error is at most 10 for \(n \geq N\).

(c) Using Eq. (3), show that for \(n \geq 100\), the th most popular website received at most 100 more visitors than the \((n + 1)\)st website.

SOLUTION

(a) In Mathematica, using the command \(\text{Sum}[10^6/n, \{n, 50\}]\) yields \(4.49921 \times 10^6\) and the command \(\text{Sum}[10^6/n, \{n, 15000\}]\) yields \(1.01931 \times 10^7\). We see that the first 50 sites get around 4.4 million hits, nearly half the 10.19 million hits of the first 15000 sites.

(b) We use \(V[n] := 10^6/n\), and compute the error \(V(n + 1) - V(n) - V'(n)\) for various values of \(n\). The table of values computed follows:

<table>
<thead>
<tr>
<th>(n)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>((V(n + 1) - V(n)) - V'(n))</td>
<td>909.091</td>
<td>119.048</td>
<td>35.8423</td>
<td>15.2489</td>
<td>7.84314</td>
</tr>
</tbody>
</table>

The error decreases in every entry. Furthermore, for \(n > 50\), the error appears to be less than 10.

(c) Since \(V(n) = 10^6n^{-1}\), \(V'(n) = -10^6n^{-2}\). The marginal derivative estimate Eq. (3) tells us that

\[ V(n) - V(n + 1) \approx -V'(n) = 10^6n^{-2}. \]

If \(n \geq 100\), \(-V'(n) \leq 10^6(100)^{-2} = 10^6(10^{-4}) = 100\). Therefore \(V(n) - V(n + 1) < 100\) for \(n \geq 100\).

In Exercises 53 and 54, the average cost per unit at production level \(x\) is defined as \(C_{\text{avg}}(x) = C(x)/x\), where \(C(x)\) is the cost function. Average cost is a measure of the efficiency of the production process.

53. Show that \(C_{\text{avg}}(x)\) is equal to the slope of the line through the origin and the point \((x, C(x))\) on the graph of \(C(x)\). Using this interpretation, determine whether average cost or marginal cost is greater at points \(A, B, C, D\) in Figure 6.

SOLUTION By definition, the slope of the line through the origin and \((x, C(x))\), that is, between \((0, 0)\) and \((x, C(x))\) is

\[ \frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{\text{av}}. \]

At point \(A\), average cost is greater than marginal cost, as the line from the origin to \(A\) is steeper than the curve at this point (we see this because the line, tracing from the origin, crosses the curve from below). At point \(B\), the average cost is still greater than the marginal cost. At the point \(C\), the average cost and the marginal cost are nearly the same, since the tangent line and the line from the origin are nearly the same. The line from the origin to \(D\) crosses the cost curve from above, and so is less steep than the tangent line to the curve at \(D\); the average cost at this point is less than the marginal cost.

54. The cost in dollars of producing alarm clocks is \(C(x) = 50x^3 - 750x^2 + 3740x + 3750\) where \(x\) is in units of 1000.

(a) Calculate the average cost at \(x = 4, 6, 8,\) and 10.

(b) Use the graphical interpretation of average cost to find the production level \(x_0\) at which average cost is lowest. What is the relation between average cost and marginal cost at \(x_0\) (see Figure 7)?
**CHAPTER 3  DIFFERENTIATION**

**SOLUTION** Let $C(x) = 50x^3 - 750x^2 + 3740x + 3750$.

(a) The slope of the line through the origin and the point $(x, C(x))$ is

\[
\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{av}(x),
\]

the average cost.

<table>
<thead>
<tr>
<th>$x$</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(x)$</td>
<td>9910</td>
<td>9990</td>
<td>11270</td>
<td>16150</td>
</tr>
<tr>
<td>$C_{av}(x)$</td>
<td>2477.5</td>
<td>1665</td>
<td>1408.75</td>
<td>1615</td>
</tr>
</tbody>
</table>

(b) The average cost is lowest at the point $P_0$ where the angle between the $x$-axis and the line through the origin and $P_0$ is lowest. This is at the point $(8, 11270)$, where the line through the origin and the graph of $C(x)$ meet in the figure above. You can see that the line is also tangent to the graph of $C(x)$, so the average cost and the marginal cost are equal at this point.

### 3.5 Higher Derivatives

**Preliminary Questions**

1. On September 4, 2003, the *Wall Street Journal* printed the headline “Stocks Go Higher, Though the Pace of Their Gains Slows.” Rephrase this headline as a statement about the first and second time derivatives of stock prices and sketch a possible graph.

**SOLUTION** Because stocks are going higher, stock prices are increasing and the first derivative of stock prices must therefore be positive. On the other hand, because the pace of gains is slowing, the second derivative of stock prices must be negative.

![Graph of stock price vs. time](image)

2. True or false? The third derivative of position with respect to time is zero for an object falling to earth under the influence of gravity. Explain.

**SOLUTION** This statement is true. The acceleration of an object falling to earth under the influence of gravity is constant; hence, the second derivative of position with respect to time is constant. Because the third derivative is just the derivative of the second derivative and the derivative of a constant is zero, it follows that the third derivative is zero.

3. Which type of polynomial satisfies $f'''(x) = 0$ for all $x$?

**SOLUTION** The third derivative of all quadratic polynomials (polynomials of the form $ax^2 + bx + c$ for some constants $a$, $b$ and $c$) is equal to 0 for all $x$.

4. What is the millionth derivative of $f(x) = e^x$?

**SOLUTION** Every derivative of $f(x) = e^x$ is $e^x$.

**Exercises**

*In Exercises 1–16, calculate $y''$ and $y'''$.*

1. $y = 14x^2$

**SOLUTION** Let $y = 14x^2$. Then $y' = 28x$, $y'' = 28$, and $y''' = 0$.

2. $y = 7 - 2x$

**SOLUTION** Let $y = 7 - 2x$. Then $y' = -2$, $y'' = 0$, and $y''' = 0$.

3. $y = x^4 - 25x^2 + 2x$

**SOLUTION** Let $y = x^4 - 25x^2 + 2x$. Then $y' = 4x^3 - 50x + 2$, $y'' = 12x^2 - 50$, and $y''' = 24x$.

4. $y = 4t^3 - 9t^2 + 7$