SOLUTION  Figure (A) satisfies the inequality
\[
\frac{f(a + h) - f(a)}{2h} \leq \frac{f(a + h) - f(a)}{h}
\]
since in this graph the symmetric difference quotient has a larger negative slope than the ordinary right difference quotient. [In figure (B), the symmetric difference quotient has a larger positive slope than the ordinary right difference quotient and therefore does not satisfy the stated inequality.]

75. Show that if \( f(x) \) is a quadratic polynomial, then the SDQ at \( x = a \) (for any \( h \neq 0 \)) is equal to \( f'(a) \). Explain the graphical meaning of this result.

SOLUTION  Let \( f(x) = px^2 + qx + r \) be a quadratic polynomial. We compute the SDQ at \( x = a \).
\[
\frac{f(a + h) - f(a - h)}{2h} = \frac{p(a + h)^2 + q(a + h) + r - (p(a - h)^2 + q(a - h) + r)}{2h}
\]
\[
= \frac{p(a^2 + 2ah + h^2) + qa + ph^2 + qa - ph^2 - a^2 - qh - r}{2h}
\]
\[
= \frac{4pa + 2qh}{2h} = \frac{2h(2pa + q)}{2h} = 2pa + q
\]

Since this doesn’t depend on \( h \), the limit, which is equal to \( f'(a) \), is also \( 2pa + q \). Graphically, this result tells us that the secant line to a parabola passing through points chosen symmetrically about \( x = a \) is always parallel to the tangent line at \( x = a \).

76. Let \( f(x) = x^{-2} \). Compute \( f'(1) \) by taking the limit of the SDQs (with \( a = 1 \)) as \( h \to 0 \).

SOLUTION  Let \( f(x) = x^{-2} \). With \( a = 1 \), the symmetric difference quotient is
\[
\frac{f(1 + h) - f(1 - h)}{2h} = \frac{\frac{1}{(1+h)^2} - \frac{1}{(1-h)^2}}{2h} = \frac{(1-h)^2 - (1+h)^2}{2h(1-h)^2(1+h)^2} = \frac{-4h}{2h(1-h)^2(1+h)^2} = -\frac{2}{(1-h)^2(1+h)^2}
\]

Therefore,
\[
f'(1) = \lim_{h \to 0} -\frac{2}{(1-h)^2(1+h)^2} = -2.
\]

3.2 The Derivative as a Function

Preliminary Questions

1. What is the slope of the tangent line through the point (2, \( f(2) \)) if \( f'(x) = x^3 \)?

SOLUTION  The slope of the tangent line through the point (2, \( f(2) \)) is given by \( f'(x) \). Since \( f'(x) = x^3 \), it follows that \( f'(2) = 2^3 = 8 \).

2. Evaluate \( (f - g)'(1) \) and \( (3f + 2g)'(1) \) assuming that \( f'(1) = 3 \) and \( g'(1) = 5 \).

SOLUTION  \( (f - g)'(1) = f'(1) - g'(1) = 3 - 5 = -2 \) and \( (3f + 2g)'(1) = 3f'(1) + 2g'(1) = 3(3) + 2(5) = 19 \).

3. To which of the following does the Power Rule apply?
   (a) \( f(x) = x^2 \)
   (b) \( f(x) = 2^x \)
   (c) \( f(x) = x^e \)
   (d) \( f(x) = e^x \)
   (e) \( f(x) = x^x \)
   (f) \( f(x) = x^{-4/5} \)

SOLUTION  (a) Yes. \( x^2 \) is a power function, so the Power Rule can be applied.
   (b) Yes. \( 2^x \) is a constant function, so the Power Rule can be applied.
   (c) Yes. \( x^e \) is a power function, so the Power Rule can be applied.
   (d) No. \( e^x \) is an exponential function (the base is constant while the exponent is a variable), so the Power Rule does not apply.
   (e) No. \( x^x \) is not a power function because both the base and the exponent are variable, so the Power Rule does not apply.
   (f) Yes. \( x^{-4/5} \) is a power function, so the Power Rule can be applied.

4. Choose (a) or (b). The derivative does not exist if the tangent line is: (a) horizontal (b) vertical.

SOLUTION  The derivative does not exist when: (b) the tangent line is vertical. At a horizontal tangent, the derivative is zero.
5. Which property distinguishes \( f(x) = e^x \) from all other exponential functions \( g(x) = b^x \)?

**SOLUTION** The line tangent to \( f(x) = e^x \) at \( x = 0 \) has slope equal to 1.

### Exercises

In Exercises 1–6, compute \( f'(x) \) using the limit definition.

1. \( f(x) = 3x - 7 \)

**SOLUTION** Let \( f(x) = 3x - 7 \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h) - 7 - (3x - 7)}{h} = \lim_{h \to 0} \frac{3h}{h} = 3.
\]

2. \( f(x) = x^2 + 3x \)

**SOLUTION** Let \( f(x) = x^2 + 3x \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^2 + 3(x + h) - (x^2 + 3x)}{h}
= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h \to 0} (2x + h + 3) = 2x + 3.
\]

3. \( f(x) = x^3 \)

**SOLUTION** Let \( f(x) = x^3 \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}
= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}
= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2.
\]

4. \( f(x) = 1 - x^{-1} \)

**SOLUTION** Let \( f(x) = 1 - x^{-1} \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1 - \frac{1}{x + h} - \left( 1 - \frac{1}{x} \right)}{h}
= \lim_{h \to 0} \frac{(x + h) - x}{x(x + h)} = \lim_{h \to 0} \frac{1}{x(x + h)} = \frac{1}{x^2}.
\]

5. \( f(x) = x - \sqrt{x} \)

**SOLUTION** Let \( f(x) = x - \sqrt{x} \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{x + h - \sqrt{x + h} - (x - \sqrt{x})}{h}
= 1 - \frac{1}{h(\sqrt{x + h} + \sqrt{x})}
= 1 - \lim_{h \to 0} \frac{1}{\sqrt{x} + h + \sqrt{x}} = 1 - \frac{1}{2\sqrt{x}}.
\]

6. \( f(x) = x^{-1/2} \)

**SOLUTION** Let \( f(x) = x^{-1/2} \). Then,

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{\sqrt{x + h} - \sqrt{x}} = \lim_{h \to 0} \frac{\sqrt{x + h} + \sqrt{x}}{h(\sqrt{x + h} + \sqrt{x})}
= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x + h}}{h(\sqrt{x} + \sqrt{x + h})}
= \lim_{h \to 0} \frac{\sqrt{x}(\sqrt{x} - 1)}{h(\sqrt{x} + \sqrt{x + h})}
= \lim_{h \to 0} \frac{\sqrt{x} - 1}{2\sqrt{x}} = \frac{-1}{2\sqrt{x}}.
\]

Multiplying the numerator and denominator of the expression by \( \sqrt{x + h} + \sqrt{x} \), we obtain:

\[
f'(x) = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x + h}}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{x - (x + h)}{h(\sqrt{x + h} + \sqrt{x})} = \lim_{h \to 0} \frac{\sqrt{x} - 1}{2\sqrt{x}} = \frac{-1}{2\sqrt{x}}.
\]
In Exercises 7–14, use the Power Rule to compute the derivative.

7. \( \frac{d}{dx} x^4 \bigg|_{x=-2} \)

**SOLUTION** \( \frac{d}{dx} (x^4) = 4x^3 \) so \( \frac{d}{dx} x^4 \bigg|_{x=-2} = 4(-2)^3 = -32. \)

8. \( \frac{d}{dt} t^{-3} \bigg|_{t=4} \)

**SOLUTION** \( \frac{d}{dt} (t^{-3}) = -3t^{-4} \) so \( \frac{d}{dt} t^{-3} \bigg|_{t=4} = -3(4)^{-4} = -\frac{3}{256}. \)

9. \( \frac{d}{dt} t^{2/3} \bigg|_{t=8} \)

**SOLUTION** \( \frac{d}{dt} (t^{2/3}) = \frac{2}{3}t^{-1/3} \) so \( \frac{d}{dt} t^{2/3} \bigg|_{t=8} = \frac{2}{3}(8)^{-1/3} = \frac{1}{3}. \)

10. \( \frac{d}{dt} t^{-2/5} \bigg|_{t=1} \)

**SOLUTION** \( \frac{d}{dt} (t^{-2/5}) = -\frac{2}{5}t^{-7/5} \) so \( \frac{d}{dt} t^{-2/5} \bigg|_{t=1} = -\frac{2}{5}(1)^{-7/5} = -\frac{2}{5}. \)

11. \( \frac{d}{dx} x^{0.35} \)

**SOLUTION** \( \frac{d}{dx} (x^{0.35}) = 0.35(x^{0.35-1}) = 0.35x^{-0.65}. \)

12. \( \frac{d}{dx} x^{14/3} \)

**SOLUTION** \( \frac{d}{dx} (x^{14/3}) = \frac{14}{3} \left( x^{(14/3)-1} \right) = \frac{14}{3} x^{11/3}. \)

13. \( \frac{d}{dt} \sqrt[3]{t} \)

**SOLUTION** \( \frac{d}{dt} \left( \sqrt[3]{t} \right) = \frac{1}{3}t^{-2/3} \)

14. \( \frac{d}{dt} (t^{-\pi^2}) \)

**SOLUTION** \( \frac{d}{dt} (t^{-\pi^2}) = -\pi^2t^{-\pi^2-1} \)

In Exercises 15–18, compute \( f'(x) \) and find an equation of the tangent line to the graph at \( x = a. \)

15. \( f(x) = x^4, \ a = 2 \)

**SOLUTION** Let \( f(x) = x^4. \) Then, by the Power Rule, \( f'(x) = 4x^3. \) The equation of the tangent line to the graph of \( f(x) \) at \( x = 2 \) is

\[ y = f'(2)(x-2) + f(2) = 32(x-2) + 16 = 32x - 48. \]

16. \( f(x) = x^{-2}, \ a = 5 \)

**SOLUTION** Let \( f(x) = x^{-2}. \) Using the Power Rule, \( f'(x) = -2x^{-3}. \) The equation of the tangent line to the graph of \( f(x) \) at \( x = 5 \) is

\[ y = f'(5)(x-5) + f(5) = -\frac{2}{125}(x-5) + \frac{1}{25} = -\frac{2}{125}x + \frac{3}{25}. \]

17. \( f(x) = 5x - 32\sqrt{x}, \ a = 4 \)

**SOLUTION** Let \( f(x) = 5x - 32x^{1/2}. \) Then \( f'(x) = 5 - 16x^{-1/2}. \) In particular, \( f'(4) = -3. \) The tangent line at \( x = 4 \) is

\[ y = f'(4)(x-4) + f(4) = -3(x-4) - 44 = -3x - 32. \]

18. \( f(x) = \sqrt[3]{x}, \ a = 8 \)

**SOLUTION** Let \( f(x) = \sqrt[3]{x} = x^{1/3}. \) Then \( f'(x) = \frac{1}{3}(x^{1/3-1}) = \frac{1}{3}x^{-2/3}. \) In particular, \( f'(8) = \frac{1}{12}. \) \( f(8) = 2, \) so the tangent line at \( x = 8 \) is

\[ y = f'(8)(x-8) + f(8) = \frac{1}{12}(x-8) + 2 = \frac{1}{12}x + \frac{4}{3}. \]
In Exercises 21–32, calculate the derivative.

21. \( f(x) = 2x^3 - 3x^2 + 5 \)

**SOLUTION**
\[
\frac{df}{dx} = 6x^2 - 6x.
\]

22. \( f(x) = 2x^3 - 3x^2 + 2x \)

**SOLUTION**
\[
\frac{df}{dx} = 6x^2 - 6x + 2.
\]

23. \( f(x) = 4x^{5/3} - 3x^{-2} - 12 \)

**SOLUTION**
\[
\frac{df}{dx} = \frac{20}{3}x^{2/3} + 6x^{-3}.
\]

24. \( f(x) = x^{5/4} + 4x^{-3/2} + 11x \)

**SOLUTION**
\[
\frac{df}{dx} = \frac{5}{4}x^{1/4} - 6x^{-5/2} + 11.
\]

25. \( g(z) = 7z^{-5/4} + z^{-5} + 9 \)

**SOLUTION**
\[
\frac{dg}{dz} = -\frac{35}{4}z^{-9/4} - 5z^{-6}.
\]

26. \( h(t) = 6 \sqrt{t} + \frac{1}{\sqrt{t}} \)

**SOLUTION**
\[
\frac{dh}{dt} = 3t^{-1/2} - \frac{1}{2}t^{-3/2}.
\]

27. \( f(s) = \sqrt[3]{s} + \sqrt[5]{s} \)

**SOLUTION**
\[
f(s) = \sqrt[3]{s} + \sqrt[5]{s} = s^{1/3} + s^{1/5}.
\]

In this form, we can apply the Sum and Power Rules.
\[
\frac{df}{ds} = \frac{1}{3}s^{-2/3} + \frac{1}{5}s^{-4/5}.
\]

28. \( W(y) = 6y^4 + 7y^{2/3} \)

**SOLUTION**
\[
\frac{dW}{dy} = 24y^3 + \frac{14}{3}y^{-1/3}.
\]

29. \( g(x) = e^2 \)

**SOLUTION**
Because \( e^2 \) is a constant, \( \frac{dg}{dx} = 0. \)

30. \( f(x) = 3e^x - x^3 \)

**SOLUTION**
\[
\frac{df}{dx} = 3e^x - 3x^2.
\]

31. \( h(t) = 5e^t - 3 \)

**SOLUTION**
\[
\frac{dh}{dt} = 5e^t - 3.
\]

\[
\frac{dh}{dt} = 5e^t - 3.
\]
32. \( f(x) = 9 - 12x^{1/3} + 8e^x \)

**SOLUTION**

\[
\frac{d}{dx} \left( 9 - 12x^{1/3} + 8e^x \right) = -4x^{-2/3} + 8e^x.
\]

In Exercises 33–36, calculate the derivative by expanding or simplifying the function.

33. \( P(s) = (4s - 3)^2 \)

**SOLUTION**

\[
P(s) = (4s - 3)^2 = 16s^2 - 24s + 9.
\]

Thus,

\[
\frac{dP}{ds} = 32s - 24.
\]

34. \( Q(r) = (1 - 2r)(3r + 5) \)

**SOLUTION**

\[
Q(r) = (1 - 2r)(3r + 5) = -6r^2 - 7r + 5.
\]

Thus,

\[
\frac{dQ}{dr} = -12r - 7.
\]

35. \( g(x) = \frac{x^2 + 4x^{1/2}}{x^2} \)

**SOLUTION**

\[
g(x) = \frac{x^2 + 4x^{1/2}}{x^2} = 1 + 4x^{-3/2}.
\]

Thus,

\[
\frac{dg}{dx} = -6x^{-5/2}.
\]

36. \( s(t) = \frac{1 - 2t}{t^{1/2}} \)

**SOLUTION**

\[
s(t) = \frac{1 - 2t}{t^{1/2}} = t^{-1/2} - 2t^{1/2}.
\]

Thus,

\[
\frac{ds}{dt} = -\frac{1}{2}t^{-3/2} - t^{-1/2}.
\]

In Exercises 37–42, calculate the derivative indicated.

37. \( \frac{dT}{dC} \bigg|_{C=8}, \quad T = 3C^{2/3} \)

**SOLUTION**

With \( T(C) = 3C^{2/3} \), we have \( \frac{dT}{dC} = 2C^{-1/3} \). Therefore,

\[
\frac{dT}{dC} \bigg|_{C=8} = 2(8)^{-1/3} = 1.
\]

38. \( \frac{dP}{dV} \bigg|_{V=-2}, \quad P = \frac{7}{V} \)

**SOLUTION**

With \( P = 7V^{-1} \), we have \( \frac{dP}{dV} = -7V^{-2} \). Therefore,

\[
\frac{dP}{dV} \bigg|_{V=-2} = -7(-2)^{-2} = -\frac{7}{4}.
\]

39. \( \frac{ds}{dz} \bigg|_{z=2}, \quad s = 4z - 16z^2 \)

**SOLUTION**

With \( s = 4z - 16z^2 \), we have \( \frac{ds}{dz} = 4 - 32z \). Therefore,

\[
\frac{ds}{dz} \bigg|_{z=2} = 4 - 32(2) = -60.
\]

40. \( \frac{dR}{dW} \bigg|_{W=1}, \quad R = W^\pi \)

**SOLUTION**

Let \( R(W) = W^\pi \). Then \( dR/dW = \pi W^{\pi - 1} \). Therefore,

\[
\frac{dR}{dW} \bigg|_{W=1} = \pi (1)^{\pi - 1} = \pi.
\]
41. \( \frac{dr}{dt} \bigg|_{t=4} = r = t - e^t \)

**SOLUTION** With \( r = t - e^t \), we have \( \frac{dr}{dt} = 1 - e^t \). Therefore,

\[
\left. \frac{dr}{dt} \right|_{t=4} = 1 - e^4.
\]

42. \( \frac{dp}{dh} \bigg|_{h=4} = p = 7e^{h-2} \)

**SOLUTION** With \( p = 7e^{h-2} \), we have \( \frac{dp}{dh} = 7e^{h-2} \). Therefore,

\[
\left. \frac{dp}{dh} \right|_{h=4} = 7e^{4-2} = 7e^2.
\]

43. Match the functions in graphs (A)–(D) with their derivatives (I)–(III) in Figure 1. Note that two of the functions have the same derivative. Explain why.

![FIGURE 1](image)

**SOLUTION**

- Consider the graph in (A). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing \( x \). This matches the graph in (III).
- Consider the graph in (B). This is a linear function, so its slope is constant. Thus the derivative is constant, which matches the graph in (I).
- Consider the graph in (C). Moving from left to right, the slope of the tangent line transitions from positive to negative then back to positive. The derivative should therefore be negative in the middle and positive to either side. This matches the graph in (II).
- Consider the graph in (D). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing \( x \). This matches the graph in (III).

Note that the functions whose graphs are shown in (A) and (D) have the same derivative. This happens because the graph in (D) is just a vertical translation of the graph in (A), which means the two functions differ by a constant. The derivative of a constant is zero, so the two functions end up with the same derivative.

44. Of the two functions \( f \) and \( g \) in Figure 2, which is the derivative of the other? Justify your answer.

![FIGURE 2](image)

**SOLUTION** \( g(x) \) is the derivative of \( f(x) \). For \( f(x) \) the slope is negative for negative values of \( x \) until \( x = 0 \), where there is a horizontal tangent, and then the slope is positive for positive values of \( x \). Notice that \( g(x) \) is negative for negative values of \( x \), goes through the origin at \( x = 0 \), and then is positive for positive values of \( x \).
45. Assign the labels \( f(x) \), \( g(x) \), and \( h(x) \) to the graphs in Figure 3 in such a way that \( f'(x) = g(x) \) and \( g'(x) = h(x) \).

![Graphs A, B, and C with labels](image)

**FIGURE 3**

**SOLUTION**  Consider the graph in (A). Moving from left to right, the slope of the tangent line is positive over the first quarter of the graph, negative in the middle half and positive again over the final quarter. The derivative of this function must therefore be negative in the middle and positive on either side. This matches the graph in (C).

Now focus on the graph in (C). The slope of the tangent line is negative over the left half and positive on the right half. The derivative of this function therefore needs to be negative on the left and positive on the right. This description matches the graph in (B).

We should therefore label the graph in (A) as \( f(x) \), the graph in (B) as \( h(x) \), and the graph in (C) as \( g(x) \). Then \( f'(x) = g(x) \) and \( g'(x) = h(x) \).

46. According to the peak oil theory, first proposed in 1956 by geophysicist M. Hubbert, the total amount of crude oil \( Q(t) \) produced worldwide up to time \( t \) has a graph like that in Figure 4.

(a) Sketch the derivative \( Q'(t) \) for \( 1900 \leq t \leq 2150 \). What does \( Q'(t) \) represent?

(b) In which year (approximately) does \( Q'(t) \) take on its maximum value?

(c) What is \( L = \lim_{t \to \infty} Q(t) \)? And what is its interpretation?

(d) What is the value of \( \lim_{t \to \infty} Q'(t) \)?

![Graph of total oil production up to time \( t \)](image)

**FIGURE 4** Total oil production up to time \( t \)

**SOLUTION**  

(a) One possible derivative sketch is shown below. Because the graph of \( Q(t) \) is roughly horizontal around \( t = 1900 \), the graph of \( Q'(t) \) begins near zero. Until roughly \( t = 2000 \), the graph of \( Q(t) \) increases more and more rapidly, so the graph of \( Q'(t) \) increases. Thereafter, the graph of \( Q(t) \) increases more and more gradually, so the graph of \( Q'(t) \) decreases. Around \( t = 2150 \), the graph of \( Q(t) \) is again roughly horizontal, so the graph of \( Q'(t) \) returns to zero. Note that \( Q'(t) \) represents the rate of change in total worldwide oil production; that is, the number of barrels produced per year.

(b) The graph of \( Q(t) \) appears to be increasing most rapidly around the year 2000, so \( Q'(t) \) takes on its maximum value around the year 2000.

(c) From Figure 4

\[ L = \lim_{t \to \infty} Q(t) = 2.3 \]

trillion barrels of oil. This value represents the total number of barrels of oil that can be produced by the planet.

(d) Because the graph of \( Q(t) \) appears to approach a horizontal line as \( t \to \infty \), it appears that

\[ \lim_{t \to \infty} Q'(t) = 0. \]
47. Use the table of values of \( f(x) \) to determine which of (A) or (B) in Figure 5 is the graph of \( f'(x) \). Explain.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>10</td>
<td>55</td>
<td>98</td>
<td>139</td>
<td>177</td>
<td>210</td>
<td>237</td>
<td>257</td>
<td>268</td>
</tr>
</tbody>
</table>

**FIGURE 5** Which is the graph of \( f'(x) \)?

**SOLUTION** The increment between successive \( x \) values in the table is a constant 0.5 but the increment between successive \( f(x) \) values decreases from 45 to 43 to 41 to 38 and so on. Thus the difference quotients decrease with increasing \( x \), suggesting that \( f'(x) \) decreases as a function of \( x \). Because the graph in (B) depicts a decreasing function, (B) might be the graph of the derivative of \( f(x) \).

48. Let \( R \) be a variable and \( r \) a constant. Compute the derivatives:

\[ \begin{align*}
&\text{(a)} \quad \frac{d}{dR} R = 1, \text{ since } R \text{ is a linear function of } R \text{ with slope 1.} \\
&\text{(b)} \quad \frac{d}{dR} r = 0, \text{ since } r \text{ is a constant.} \\
&\text{(c)} \quad \frac{d}{dR} r^2R^3 = r^2 \frac{d}{dR} R^3 = r^2(3R^2) = 3r^2R^2.
\end{align*} \]

**SOLUTION**

\[ \begin{align*}
&\text{(a)} \quad \frac{d}{dt} ct^3 = 3ct^2. \\
&\text{(b)} \quad \frac{d}{dz} (5z + 4ez^2) = 5 + 8ez. \\
&\text{(c)} \quad \frac{d}{dy} (9c^3y^3 - 24c) = 27c^2y^2.
\end{align*} \]

49. Compute the derivatives, where \( c \) is a constant.

\[ \begin{align*}
&\text{(a)} \quad \frac{d}{dt} ct^3 = 3ct^2. \\
&\text{(b)} \quad \frac{d}{dz} (5z + 4ez^2) = 5 + 8ez. \\
&\text{(c)} \quad \frac{d}{dy} (9c^3y^3 - 24c) = 27c^2y^2.
\end{align*} \]

50. Find the points on the graph of \( f(x) = 12x - x^3 \) where the tangent line is horizontal.

**SOLUTION** Let \( f(x) = 12x - x^3 \). Solve \( f'(x) = 12 - 2x^2 = 0 \) to obtain \( x = \pm \sqrt{6} \). Thus, the graph of \( f(x) = 12x - x^3 \) has a horizontal tangent line at two points: \((\sqrt{6}, 6\sqrt{6})\) and \((-\sqrt{6}, -6\sqrt{6})\).

51. Find the points on the graph of \( y = x^2 + 3x - 7 \) at which the slope of the tangent line is equal to 4.

**SOLUTION** Let \( y = x^2 + 3x - 7 \). Solving \( dy/dx = 2x + 3 = 4 \) yields \( x = \frac{1}{2} \).

52. Find the values of \( x \) where \( y = x^3 \) and \( y = x^2 + 5x \) have parallel tangent lines.

**SOLUTION** Let \( f(x) = x^3 \) and \( g(x) = x^2 + 5x \). The graphs have parallel tangent lines when \( f'(x) = g'(x) \). Hence, we solve \( f'(x) = 3x^2 = 2x + 5 = g'(x) \) to obtain \( x = \frac{5}{3} \) and \( x = -1 \).

53. Determine \( a \) and \( b \) such that \( p(x) = x^2 + ax + b \) satisfies \( p(1) = 0 \) and \( p'(1) = 4 \).

**SOLUTION** Let \( p(x) = x^2 + ax + b \) satisfy \( p(1) = 0 \) and \( p'(1) = 4 \). Now, \( p'(x) = 2x + a \). Therefore \( 0 = p(1) = 1 + a + b \) and \( 4 = p'(1) = 2 + a; i.e., a = 2 and b = -3 \).

54. Find all values of \( x \) such that the tangent line to \( y = 4x^2 + 11x + 2 \) is steeper than the tangent line to \( y = x^3 \).
SOLUTION Let \( f(x) = 4x^2 + 11x + 2 \) and let \( g(x) = x^3 \). We need all \( x \) such that \( f'(x) > g'(x) \).

\[
f'(x) > g'(x)
\]
\[
8x + 11 > 3x^2
\]
\[
0 > 3x^2 - 8x - 11
\]
\[
0 > (3x - 11)(x + 1).
\]
The product \((3x - 11)(x + 1) = 0\) when \( x = -1 \) and when \( x = \frac{11}{3} \). We therefore examine the intervals \(-1 < x < \frac{11}{3} \) and \( x > \frac{11}{3} \). For \( x < -1 \) and for \( x > \frac{11}{3} \), \((3x - 11)(x + 1) > 0\), whereas for \(-1 < x < \frac{11}{3} \), \((3x - 11)(x + 1) < 0\). The solution set is therefore \(-1 < x < \frac{11}{3} \).

55. Let \( f(x) = x^3 - 3x + 1 \). Show that \( f'(x) \geq -3 \) for all \( x \) and that, for every \( m > -3 \), there are precisely two points where \( f'(x) = m \). Indicate the position of these points and the corresponding tangent lines for one value of \( m \) in a sketch of the graph of \( f(x) \).

SOLUTION Let \( P = (a, b) \) be a point on the graph of \( f(x) = x^3 - 3x + 1 \).

- The derivative satisfies \( f'(x) = 3x^2 - 3 \geq -3 \) since \( 3x^2 \) is nonnegative.
- Suppose the slope \( m \) of the tangent line is greater than \(-3 \). Then \( f'(a) = 3a^2 - 3 = m \), whence

\[
a^2 = \frac{m + 3}{3} > 0 \quad \text{and thus} \quad a = \pm \sqrt[3]{\frac{m + 3}{3}}.
\]
- The two parallel tangent lines with slope 2 are shown with the graph of \( f(x) \) here.

\[ \begin{array}{c}
\text{Graph of } f(x) = x^3 - 3x + 1
\end{array} \]

56. Show that the tangent lines to \( y = \frac{1}{4}x^3 - x^2 \) at \( x = a \) and at \( x = b \) are parallel if \( a = b \) or \( a + b = 2 \).

SOLUTION Let \( P = (a, f(a)) \) and \( Q = (b, f(b)) \) be points on the graph of \( y = f(x) = \frac{1}{4}x^3 - x^2 \). Equate the slopes of the tangent lines at the points \( P \) and \( Q \):

\[
a^2 - 2a - b^2 + 2b = (a - b)(a + b) - 2(a - b) = (a - 2 + b)(a - b);
\]

therefore, either \( a = b \) (i.e., \( P \) and \( Q \) are the same point) or \( a + b = 2 \).

57. Compute the derivative of \( f(x) = x^{3/2} \) using the limit definition. Hint: Show that

\[
\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^{3/2} - x^{3/2}}{h} \left( \frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}} \right)
\]

SOLUTION Once we have the difference of square roots, we multiply by the conjugate to solve the problem.

\[
f'(x) = \lim_{h \to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h} \left( \frac{\sqrt{(x+h)^3} + \sqrt{x^3}}{\sqrt{(x+h)^3} + \sqrt{x^3}} \right)
\]

\[
= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \left( \frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}} \right).
\]

The first factor of the expression in the last line is clearly the limit definition of the derivative of \( x^3 \), which is \( 3x^2 \). The second factor can be evaluated, so

\[
\frac{d}{dx} x^{3/2} = 3x^{2/2} \frac{1}{2\sqrt{x^3}} = \frac{3}{2} x^{1/2}.
\]

58. Use the limit definition of \( m(h) \) to approximate \( m(4) \). Then estimate the slope of the tangent line to \( y = 4^x \) at \( x = 0 \) and \( x = 2 \).

SOLUTION Recall

\[
m(4) = \lim_{h \to 0} \left( \frac{4^h - 1}{h} \right).
\]
Using a table of values, we find

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\frac{4h - 1}{h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.39595</td>
</tr>
<tr>
<td>0.001</td>
<td>1.38726</td>
</tr>
<tr>
<td>0.0001</td>
<td>1.38639</td>
</tr>
<tr>
<td>0.00001</td>
<td>1.38630</td>
</tr>
</tbody>
</table>

Thus $m(4) \approx 1.386$. Knowing that $y'(x) = m(4) \cdot 4^x$, it follows that $y'(0) \approx 1.386$ and $y'(2) \approx 1.386 \cdot 16 = 22.176$.

59. Let $f(x) = xe^x$. Use the limit definition to compute $f'(0)$, and find the equation of the tangent line at $x = 0$.

**SOLUTION** Let $f(x) = xe^x$. Then $f(0) = 0$, and

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{he^h - 0}{h} = \lim_{h \to 0} e^h = 1.$$  

The equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = 1(x - 0) + 0 = x.$$

60. The average speed (in meters per second) of a gas molecule is

$$v_{\text{avg}} = \sqrt[\frac{8RT}{\pi M}}$$

where $T$ is the temperature (in kelvins), $M$ is the molar mass (in kilograms per mole), and $R = 8.31$. Calculate $dv_{\text{avg}}/dT$ at $T = 300$ K for oxygen, which has a molar mass of 0.032 kg/mol.

**SOLUTION** Using the form $v_{\text{avg}} = (\frac{8RT}{\pi M})^{1/2} = \sqrt[1/2]{\frac{8R}{\pi M}T^{1/2}}$, where $M$ and $R$ are constant, we use the Power Rule to compute the derivative $dv_{\text{avg}}/dT$.

$$\frac{d}{dT} \sqrt[1/2]{\frac{8R}{\pi M}T^{1/2}} = \sqrt[1/2]{\frac{8R}{\pi M}} \frac{d}{dT} T^{1/2} = \sqrt[1/2]{\frac{8R}{\pi M}} \frac{1}{2} T^{(1/2)-1}.$$  

In particular, if $T = 300$ K,

$$\frac{d}{dT} v_{\text{avg}} = \sqrt{\frac{8(8.31)}{(\pi(0.032))^{1/2}}} \frac{1}{2} (300)^{-1/2} = 0.74234 \text{ m/(s \cdot K)}.$$

61. Biologists have observed that the pulse rate $P$ (in beats per minute) in animals is related to body mass (in kilograms) by the approximate formula $P = 200m^{-1/4}$. This is one of many allometric scaling laws prevalent in biology. Is $|dP/dm|$ an increasing or decreasing function of $m$? Find an equation of the tangent line at the points on the graph in Figure 6 that represent goat ($m = 33$) and man ($m = 68$).

**FIGURE 6**

![Graph showing the relationship between mass and pulse rate](image)

**SOLUTION** $dP/dm = -50m^{-5/4}$. For $m > 0$, $|dP/dm| = |50m^{-5/4}|$. $|dP/dm| \to 0$ as $m$ gets larger; $|dP/dm|$ gets smaller as $m$ gets bigger.

For each $m = c$, the equation of the tangent line to the graph of $P$ at $m$ is

$$y = P'(c)(m - c) + P(c).$$

For a goat ($m = 33$ kg), $P(33) = 83.445$ beats per minute (bpm) and

$$\frac{dP}{dm} = -50(33)^{-5/4} \approx -0.63216 \text{ bpm/kg}.$$  

Hence, $y = -0.63216(m - 33) + 83.445$.  

For a man \((m = 68 \text{ kg})\), we have \(P(68) = 69.647 \text{ bpm}\) and

\[
\frac{dP}{dm} = -50(68)^{-2/4} \approx -0.25606 \text{ bpm/kg}.
\]

Hence, the tangent line has formula \(y = -0.25606(m - 68) + 69.647\).

**62.** Some studies suggest that kidney mass \(K\) in mammals (in kilograms) is related to body mass \(m\) (in kilograms) by the approximate formula \(K = 0.007m^{0.85}\). Calculate \(dK/dm\) at \(m = 68\). Then calculate the derivative with respect to \(m\) of the relative kidney-to-mass ratio \(K/m\) at \(m = 68\).

**SOLUTION**

\[
\frac{dK}{dm} = 0.007(0.85)m^{-0.15} = 0.00595m^{-0.15};
\]

hence,

\[
\frac{dK}{dm}\bigg|_{m=68} = 0.00595(68)^{-0.15} = 0.00315966.
\]

Because

\[
\frac{K}{m} = 0.007m^{0.85}m^{-1.15} = 0.007m^{-0.15},
\]

we find

\[
\frac{d}{dm} \left( \frac{K}{m} \right) = 0.007 \frac{d}{dm} m^{-0.15} = -0.00105m^{-1.15},
\]

and

\[
\frac{d}{dm} \left( \frac{K}{m} \right)\bigg|_{m=68} = -8.19981 \times 10^{-6} \text{ kg}^{-1}.
\]

**63.** The Clausius–Clapeyron Law relates the vapor pressure of water \(P\) (in atmospheres) to the temperature \(T\) (in kelvins):

\[
\frac{dP}{dT} = k \frac{P}{T^2}
\]

where \(k\) is a constant. Estimate \(dP/dT\) for \(T = 303, 313, 323, 333, 343\) using the data and the approximation

\[
\frac{dP}{dT} \approx \frac{P(T + 10) - P(T - 10)}{20}
\]

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>293</th>
<th>303</th>
<th>313</th>
<th>323</th>
<th>333</th>
<th>343</th>
<th>353</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P) (atm)</td>
<td>0.0278</td>
<td>0.0482</td>
<td>0.0808</td>
<td>0.1311</td>
<td>0.2067</td>
<td>0.3173</td>
<td>0.4754</td>
</tr>
</tbody>
</table>

Do your estimates seem to confirm the Clausius–Clapeyron Law? What is the approximate value of \(k\)?

**SOLUTION** Using the indicated approximation to the first derivative, we calculate

\[
P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};
\]

\[
P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};
\]

\[
P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};
\]

\[
P'(333) \approx \frac{P(343) - P(333)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};
\]

\[
P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}
\]

If the Clausius–Clapeyron law is valid, then \(\frac{T^2}{P} \frac{dP}{dT}\) should remain constant as \(T\) varies. Using the data for vapor pressure and temperature and the approximate derivative values calculated above, we find

<table>
<thead>
<tr>
<th>(T) (K)</th>
<th>303</th>
<th>313</th>
<th>323</th>
<th>333</th>
<th>343</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{T^2}{P} \frac{dP}{dT})</td>
<td>5047.59</td>
<td>5025.76</td>
<td>5009.54</td>
<td>4994.57</td>
<td>4981.45</td>
</tr>
</tbody>
</table>

These values are roughly constant, suggesting that the Clausius–Clapeyron law is valid, and that \(k \approx 5000\).
64. Let $L$ be the tangent line to the hyperbola $xy = 1$ at $x = a$, where $a > 0$. Show that the area of the triangle bounded by $L$ and the coordinate axes does not depend on $a$.

**SOLUTION** Let $f(x) = x^{-1}$. The tangent line to $f$ at $x = a$ is $y = f'(a)(x - a) + f(a) = -\frac{1}{a^2}(x - a) + \frac{1}{a}$. The $y$-intercept of this line (where $x = 0$) is $\frac{2}{a}$. Its $x$-intercept (where $y = 0$) is $2a$. Hence the area of the triangle bounded by the tangent line and the coordinate axes is $A = \frac{1}{2}bh = \frac{1}{2}(2a) \left( \frac{2}{a} \right) = 2$, which is independent of $a$.

65. In the setting of Exercise 64, show that the point of tangency is the midpoint of the segment of $L$ lying in the first quadrant.

**SOLUTION** In the previous exercise, we saw that the tangent line to the hyperbola $xy = 1$ or $y = \frac{1}{x}$ at $x = a$ has $y$-intercept $P = (0, \frac{2}{a})$ and $x$-intercept $Q = (2a, 0)$. The midpoint of the line segment connecting $P$ and $Q$ is thus
\[
\left( \frac{0 + 2a}{2}, \frac{2}{a} + 0 \right) = \left( a, \frac{1}{a} \right),
\]
which is the point of tangency.

66. Match functions (A)–(C) with their derivatives (I)–(III) in Figure 7.

**SOLUTION** Note that the graph in (A) has three locations with a horizontal tangent line. The derivative must therefore cross the $x$-axis in three locations, which matches (III).

The graph in (B) has only one location with a horizontal tangent line, so its derivative should cross the $x$-axis only once. Thus, (I) is the graph corresponding to the derivative of (B).

Finally, the graph in (B) has two locations with a horizontal tangent line, so its derivative should cross the $x$-axis twice. Thus, (II) is the graph corresponding to the derivative of (C).

67. Make a rough sketch of the graph of the derivative of the function in Figure 8(A).

**FIGURE 7**

**FIGURE 8**
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SOLUTION  The graph has a tangent line with negative slope approximately on the interval (1, 3.6), and has a tangent line with a positive slope elsewhere. This implies that the derivative must be negative on the interval (1, 3.6) and positive elsewhere. The graph may therefore look like this:

68. Graph the derivative of the function in Figure 8(B), omitting points where the derivative is not defined.

SOLUTION  On (−1, 0), the graph is a line with slope −3, so the derivative is equal to −3. The derivative on (0, 2) is x. Finally, on (2, 4) the function is a line with slope −1, so the derivative is equal to −1. Combining this information leads to the graph:

69. Sketch the graph of \( f(x) = x|x| \). Then show that \( f'(0) \) exists.

SOLUTION  For \( x < 0 \), \( f(x) = −x^2 \), and \( f'(x) = −2x \). For \( x > 0 \), \( f(x) = x^2 \), and \( f'(x) = 2x \). At \( x = 0 \), we find

\[
\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0
\]

and

\[
\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0.
\]

Because the two one-sided limits exist and are equal, it follows that \( f'(0) \) exists and is equal to zero. Here is the graph of \( f(x) = x|x| \).

70. Determine the values of \( x \) at which the function in Figure 9 is: (a) discontinuous, and (b) nondifferentiable.

SOLUTION  The function is discontinuous at those points where it is undefined or there is a break in the graph. On the interval [0, 4], there is only one such point, at \( x = 1 \).

The function is nondifferentiable at those points where it is discontinuous or where it has a corner or cusp. In addition to the point \( x = 1 \) we already know about, the function is nondifferentiable at \( x = 2 \) and \( x = 3 \).
In Exercises 71–76, find the points c (if any) such that \( f'(c) \) does not exist.

71. \( f(x) = |x - 1| \)

**SOLUTION**

Here is the graph of \( f(x) = |x - 1| \). Its derivative does not exist at \( x = 1 \). At that value of \( x \) there is a sharp corner.

72. \( f(x) = [x] \)

**SOLUTION**

Here is the graph of \( f(x) = [x] \). This is the integer step function graph. Its derivative does not exist at all \( x \) values that are integers. At those values of \( x \) the graph is discontinuous.

73. \( f(x) = x^{2/3} \)

**SOLUTION**  Here is the graph of \( f(x) = x^{2/3} \). Its derivative does not exist at \( x = 0 \). At that value of \( x \), there is a sharp corner or “cusp”.

74. \( f(x) = x^{3/2} \)

**SOLUTION**  The function is differentiable on its entire domain, \( \{x : x \geq 0\} \). The formula is \( \frac{d}{dx} x^{3/2} = \frac{3}{2} x^{1/2} \).

75. \( f(x) = |x^2 - 1| \)

**SOLUTION**  Here is the graph of \( f(x) = |x^2 - 1| \). Its derivative does not exist at \( x = -1 \) or at \( x = 1 \). At these values of \( x \), the graph has sharp corners.

76. \( f(x) = |x - 1|^2 \)

**SOLUTION**

This is the graph of \( f(x) = |x - 1|^2 \). Its derivative exists everywhere.
In Exercises 77–82, zoom in on a plot of \( f(x) \) at the point \((a, f(a))\) and state whether or not \( f(x) \) appears to be differentiable at \( x = a \). If it is non-differentiable, state whether the tangent line appears to be vertical or does not exist.

77. \( f(x) = (x - 1)|x|, \quad a = 0 \)

**Solution** The graph of \( f(x) = (x - 1)|x| \) for \( x \) near 0 is shown below. Because the graph has a sharp corner at \( x = 0 \), it appears that \( f \) is not differentiable at \( x = 0 \). Moreover, the tangent line does not exist at this point.

![Graph of f(x) = (x - 1)|x|](image)

78. \( f(x) = (x - 3)^{5/3}, \quad a = 3 \)

**Solution** The graph of \( f(x) = (x - 3)^{5/3} \) for \( x \) near 3 is shown below. From this graph, it appears that \( f \) is differentiable at \( x = 3 \), with a horizontal tangent line.

![Graph of f(x) = (x - 3)^{5/3}](image)

79. \( f(x) = (x - 3)^{1/3}, \quad a = 3 \)

**Solution** The graph of \( f(x) = (x - 3)^{1/3} \) for \( x \) near 3 is shown below. From this graph, it appears that \( f \) is not differentiable at \( x = 3 \). Moreover, the tangent line appears to be vertical.

![Graph of f(x) = (x - 3)^{1/3}](image)

80. \( f(x) = \sin(x^{1/3}), \quad a = 0 \)

**Solution** The graph of \( f(x) = \sin(x^{1/3}) \) for \( x \) near 0 is shown below. From this graph, it appears that \( f \) is not differentiable at \( x = 0 \). Moreover, the tangent line appears to be vertical.

![Graph of f(x) = \sin(x^{1/3})](image)

81. \( f(x) = |\sin x|, \quad a = 0 \)

**Solution** The graph of \( f(x) = |\sin x| \) for \( x \) near 0 is shown below. Because the graph has a sharp corner at \( x = 0 \), it appears that \( f \) is not differentiable at \( x = 0 \). Moreover, the tangent line does not exist at this point.

![Graph of f(x) = |\sin x|](image)

82. \( f(x) = |x - \sin x|, \quad a = 0 \)

**Solution** The graph of \( f(x) = |x - \sin x| \) for \( x \) near 0 is shown below. From this graph, it appears that \( f \) is differentiable at \( x = 0 \), with a horizontal tangent line.

![Graph of f(x) = |x - \sin x|](image)
83. Plot the derivative $f'(x)$ of $f(x) = 2x^3 - 10x^{-1}$ for $x > 0$ (set the bounds of the viewing box appropriately) and observe that $f'(x) > 0$. What does the positivity of $f'(x)$ tell us about the graph of $f(x)$ itself? Plot $f(x)$ and confirm this conclusion.

**Solution** Let $f(x) = 2x^3 - 10x^{-1}$. Then $f'(x) = 6x^2 + 10x^{-2}$. The graph of $f'(x)$ is shown in the figure below at the left and it is clear that $f'(x) > 0$ for all $x > 0$. The positivity of $f'(x)$ tells us that the graph of $f(x)$ is increasing for $x > 0$. This is confirmed in the figure below at the right, which shows the graph of $f(x)$.

84. Find the coordinates of the point $P$ in Figure 10 at which the tangent line passes through $(5, 0)$.

**Solution** Let $f(x) = 9 - x^2$, and suppose $P$ has coordinates $(a, 9 - a^2)$. Because $f'(x) = -2x$, the slope of the line tangent to the graph of $f(x)$ at $P$ is $-2a$, and the equation of the tangent line is

$$y = f'(a)(x - a) + f(a) = -2a(x - a) + 9 - a^2 = -2ax + 9 + a^2.$$ 

In order for this line to pass through the point $(5, 0)$, we must have

$$0 = -10a + 9 + a^2 = (a - 9)(a - 1).$$

Thus, $a = 1$ or $a = 9$. We exclude $a = 9$ because from Figure 10 we are looking for an $x$-coordinate between 0 and 5. Thus, the point $P$ has coordinates $(1, 8)$.

Exercises 85–88 refer to Figure 11. Length $QR$ is called the subtangent at $P$, and length $RT$ is called the subnormal.
85. Calculate the subtangent of

\[ f(x) = x^2 + 3x \] at \( x = 2 \)

**SOLUTION** Let \( f(x) = x^2 + 3x \). Then \( f'(x) = 2x + 3 \), and the equation of the tangent line at \( x = 2 \) is

\[ y = f'(2)(x - 2) + f(2) = 7(x - 2) + 10 = 7x - 4. \]

This line intersects the x-axis at \( x = \frac{4}{7} \). Thus \( Q \) has coordinates \( (\frac{4}{7}, 0) \), \( R \) has coordinates \( (2, 0) \) and the subtangent is

\[ 2 - \frac{4}{7} = \frac{10}{7}. \]

86. Show that the subtangent of \( f(x) = e^x \) is everywhere equal to 1.

**SOLUTION** Let \( f(x) = e^x \). Then \( f'(x) = e^x \), and the equation of the tangent line at \( x = a \) is

\[ y = f'(a)(x - a) + f(a) = e^a(x - a) + e^a. \]

This line intersects the x-axis at \( x = a - 1 \). Thus, \( Q \) has coordinates \( (a - 1, 0) \), \( R \) has coordinates \( (a, 0) \) and the subtangent is

\[ a - (a - 1) = 1. \]

87. Prove in general that the subnormal at \( P \) is \( |f'(x)|f(x) \).

**SOLUTION** The slope of the tangent line at \( P \) is \( f'(x) \). The slope of the line normal to the graph at \( P \) is then \(-1/f'(x)\), and the normal line intersects the x-axis at the point \( T \) with coordinates \((x + f(x)f'(x), 0)\). The point \( R \) has coordinates \((x, 0)\), so the subnormal is

\[ |x + f(x)f'(x) - x| = |f(x)f'(x)|. \]

88. Show that \( PQ \) has length \( |f(x)|\sqrt{1 + f'(x)^2} \).

**SOLUTION** The coordinates of the point \( P \) are \((x, f(x))\), the coordinates of the point \( R \) are \((x, 0)\) and the coordinates of the point \( Q \) are

\[ \left( x - \frac{f(x)}{f'(x)}, 0 \right). \]

Thus, \( PR = |f(x)| \), \( QR = \left| \frac{f(x)}{f'(x)} \right| \), and by the Pythagorean Theorem

\[ PQ = \sqrt{\left( \frac{f(x)}{f'(x)} \right)^2 + (f(x))^2} = |f(x)|\sqrt{1 + f'(x)^2}. \]

89. Prove the following theorem of Apollonius of Perga (the Greek mathematician born in 262 BCE who gave the parabola, ellipse, and hyperbola their names): The subtangent of the parabola \( y = x^2 \) at \( x = a \) is equal to \( a/2 \).

**SOLUTION** Let \( f(x) = x^2 \). The tangent line to \( f \) at \( x = a \) is

\[ y = f'(a)(x - a) + f(a) = 2a(x - a) + a^2 = 2ax - a^2. \]

The x-intercept of this line (where \( y = 0 \)) is \( \frac{a}{2} \) as claimed.

![Diagram](https://example.com/diagram.png)

90. Show that the subtangent to \( y = x^3 \) at \( x = a \) is equal to \( \frac{1}{3}a \).

**SOLUTION** Let \( f(x) = x^3 \). Then \( f'(x) = 3x^2 \), and the equation of the tangent line at \( x = a \) is

\[ y = f'(a)(x - a) + f(a) = 3a^2(x - a) + a^3 = 3a^2x - 2a^3. \]

This line intersects the x-axis at \( x = 2a/3 \). Thus, \( Q \) has coordinates \((2a/3, 0)\), \( R \) has coordinates \((a, 0)\) and the subtangent is

\[ a - \frac{2}{3}a = \frac{1}{3}a. \]
91. Formulate and prove a generalization of Exercise 90 for $y = x^n$.

**SOLUTION** Let $f(x) = x^n$. Then $f'(x) = nx^{n-1}$, and the equation of the tangent line $tx = a$ is

$$y = f'(a)(x-a) + f(a) = na^{n-1}(x-a) + a^n = na^{n-1}x - (n-1)a^n.$$ 

This line intersects the $x$-axis at $x = (n-1)a/n$. Thus, $Q$ has coordinates $((n-1)a/n, 0)$, $R$ has coordinates $(a, 0)$ and the subtangent is

$$a - n - 1 = \frac{1}{n} n.$$ 

**Further Insights and Challenges**

92. Two small arches have the shape of parabolas. The first is given by $f(x) = 1 - x^2$ for $-1 \leq x \leq 1$ and the second by $g(x) = 4 - (x - 4)^2$ for $2 \leq x \leq 6$. A board is placed on top of these arches so it rests on both (Figure 12). What is the slope of the board?

**Hint:** Find the tangent line to $y = f(x)$ that intersects $y = g(x)$ in exactly one point.

**SOLUTION** At the points where the board makes contact with the arches the slope of the board must be equal to the slope of the arches (and hence they are equal to each other). Suppose $(t, f(t))$ is the point where the board touches the left hand arch. The tangent line here (the line the board defines) is given by

$$y = f'(t)(x-t) + f(t).$$ 

This line must hit the other arch in exactly one point. In other words, if we plug in $y = g(x)$ to get

$$g(x) = f'(t)(x-t) + f(t)$$

there can only be one solution for $x$ in terms of $t$. Computing $f'$ and plugging in we get

$$4 - (x^2 - 8x + 16) = -2tx + 2t^2 + 1 - t^2$$

which simplifies to

$$x^2 - 2tx - 8x + t^2 + 13 = 0.$$ 

This is a quadratic equation $ax^2 + bx + c = 0$ with $a = 1, b = (-2t - 8)$ and $c = t^2 + 13$. By the quadratic formula we know there is a unique solution for $x$ iff $b^2 - 4ac = 0$. In our case this means

$$(2t + 8)^2 = 4(t^2 + 13).$$

Solving this gives $t = -3/8$ and plugging into $f'$ shows the slope of the board must be $3/4$.

93. A vase is formed by rotating $y = x^2$ around the $y$-axis. If we drop in a marble, it will either touch the bottom point of the vase or be suspended above the bottom by touching the sides (Figure 13). How small must the marble be to touch the bottom?
SOLUTION Suppose a circle is tangent to the parabola \( y = x^2 \) at the point \((t, t^2)\). The slope of the parabola at this point is \(2t\), so the slope of the radius of the circle at this point is \(-\frac{1}{2t}\) (since it is perpendicular to the tangent line of the circle). Thus the center of the circle must be where the line given by \( y = -\frac{1}{2t}(x - t) + t^2 \) crosses the y-axis. We can find the y-coordinate by setting \( x = 0 \): we get \( y = \frac{1}{4} + t^2 \). Thus, the radius extends from \((0, \frac{1}{4} + t^2)\) to \((t, t^2)\) and
\[
r = \sqrt{\left(\frac{1}{2} + t^2 - t^2\right)^2 + t^2} = \sqrt{\frac{1}{4} + t^2}.
\]
This radius is greater than \(\frac{1}{2}\) whenever \(t > 0\); so, if a marble has radius \(> 0.5\) it sits on the edge of the vase, but if it has radius \(\leq 0.5\) it rolls all the way to the bottom.

94. Let \(f(x)\) be a differentiable function, and set \(g(x) = f(x + c)\), where \(c\) is a constant. Use the limit definition to show that \(g'(x) = f'(x + c)\). Explain this result graphically, recalling that the graph of \(g(x)\) is obtained by shifting the graph of \(f(x)\) \(+c\) units to the left (if \(c > 0\)) or right (if \(c < 0\)).

SOLUTION

- Let \(g(x) = f(x + c)\). Using the limit definition,
\[
g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h \to 0} \frac{f((x + h) + c) - f(x + c)}{h}
= \lim_{h \to 0} \frac{f((x + c) + h) - f(x + c)}{h} = f'(x + c).
\]

- The graph of \(g(x)\) is obtained by shifting \(f(x)\) to the left by \(c\) units. This implies that \(g'(x)\) is equal to \(f'(x + c)\) shifted to the left by \(c\) units, which happens to be \(f'(x + c)\). Therefore, \(g'(x) = f'(x + c)\).

95. Negative Exponents Let \(n\) be a whole number. Use the Power Rule for \(x^n\) to calculate the derivative of \(f(x) = x^{-n}\) by showing that
\[
f(x + h) - f(x) = \frac{-1}{x^n(x + h)^n} (x + h)^n - x^n
\]

SOLUTION Let \(f(x) = x^{-n}\) where \(n\) is a positive integer.

- The difference quotient for \(f\) is
\[
f(x + h) - f(x) = \frac{(x + h)^n - x^n}{(x + h)^n} - \frac{1}{x^n} \frac{x^n - (x + h)^n}{h}
= \frac{1}{x^n(x + h)^n} (x + h)^n - x^n
= \frac{-1}{x^n(x + h)^n} (x + h)^n - x^n.
\]

- Therefore,
\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{-1}{x^n(x + h)^n} (x + h)^n - x^n
= \lim_{h \to 0} \frac{-1}{x^n(x + h)^n} \lim_{h \to 0} (x + h)^n - x^n = -x^{-2n} \frac{d}{dx} (x^n).
\]

- From above, we continue: \(f'(x) = -x^{-2n} \frac{d}{dx} (x^n) = -x^{-2n} \cdot nx^{n-1} = nx^{-n-1}\). Since \(n\) is a positive integer, \(k = -n\) is a negative integer and we have \(\frac{d}{dx} (x^k) = \frac{d}{dx} (x^{-n}) = -nx^{-n-1} = kx^{k-1}\); i.e. \(\frac{d}{dx} (x^k) = kx^{k-1}\) for negative integers \(k\).

96. Verify the Power Rule for the exponent \(1/n\), where \(n\) is a positive integer, using the following trick: Rewrite the difference quotient for \(y = x^{1/n}\) at \(x = b\) in terms of \(u = (b + h)^{1/n}\) and \(a = b^{1/n}\).

SOLUTION Substituting \(x = (b + h)^{1/n}\) and \(a = b^{1/n}\) into the left-hand side of equation (3) yields
\[
\frac{x^n - a^n}{x - a} = \frac{(b + h)^{1/n} - b}{(b + h)^{1/n} - b^{1/n}} = \frac{h}{(b + h)^{1/n} - b^{1/n}}
\]
whereas substituting these same expressions into the right-hand side of equation (3) produces
\[
\frac{x^n - a^n}{x - a} = (b + h)^{1/n}a + (b + h)^{\frac{1}{m}}b^{1/n} + (b + h)^{\frac{2}{m}}b^{2/n} + \cdots + b^{\frac{m}{m}},
\]
hence,
\[
(b + h)^{1/n} - b^{1/n} = \frac{1}{(b + h)^{n - 1} + (b + h)^{n - 2}b^{1/n} + (b + h)^{n - 3}b^{2/n} + \ldots + b^{n - 1}}.
\]

If we take \( f(x) = x^{1/n} \), then, using the previous expression,
\[
f'(b) = \lim_{h \to 0} \frac{(b + h)^{1/n} - b^{1/n}}{h} = \frac{1}{n b^{n - 1}} = \frac{1}{n b^{n-1}}.
\]
Replacing \( b \) by \( x \), we have \( f'(x) = \frac{1}{n} x^{\frac{1}{n} - 1} \).

**97. Infinitely Rapid Oscillations**

Define
\[
f(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]

Show that \( f(x) \) is continuous at \( x = 0 \) but \( f'(0) \) does not exist (see Figure 12).

**SOLUTION** Let \( f(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases} \) as \( x \to 0 \),
\[
|f(x) - f(0)| = \left| x \sin \left( \frac{1}{x} \right) - 0 \right| = |x| \left| \sin \left( \frac{1}{x} \right) \right| \to 0
\]

since the values of the sine lie between \(-1\) and \(1\). Hence, by the Squeeze Theorem, \( \lim_{x \to 0} f(x) = f(0) \) and thus \( f \) is continuous at \( x = 0 \).

As \( x \to 0 \), the difference quotient at \( x = 0 \),
\[
\frac{f(x) - f(0)}{x - 0} = \frac{x \sin \left( \frac{1}{x} \right) - 0}{x - 0} = \sin \left( \frac{1}{x} \right)
\]
does not converge to a limit since it oscillates infinitely through every value between \(-1\) and \(1\). Accordingly, \( f'(0) \) does not exist.

**98.** For which value of \( \lambda \) does the equation \( e^x = \lambda x \) have a unique solution? For which values of \( \lambda \) does it have at least one solution? For intuition, plot \( y = e^x \) and the line \( y = \lambda x \).

**SOLUTION** First, note that when \( \lambda = 0 \), the equation \( e^x = 0 \cdot x = 0 \) has no real solution. For \( \lambda \neq 0 \), we observe that solutions to the equation \( e^x = \lambda x \) correspond to points of intersection between the graphs of \( y = e^x \) and \( y = \lambda x \). When \( \lambda < 0 \), the two graphs intersect at only one location (see the graph below at the left). On the other hand, when \( \lambda > 0 \), the graphs may have zero, one or two points of intersection (see the graph below at the right). Note that the graphs have one point of intersection when \( y = \lambda x \) is the tangent line to \( y = e^x \). Thus, not only do we require \( e^x = \lambda x \), but also \( e^x = \lambda \). It then follows that the point of intersection satisfies \( \lambda = \lambda x \), so \( x = 1 \). This then gives \( \lambda = e \).

Therefore the equation \( e^x = \lambda x \):

(a) has at least one solution when \( \lambda < 0 \) and when \( \lambda \geq e \);

(b) has a unique solution when \( \lambda < 0 \) and when \( \lambda = e \).